

The K_4 -free process

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Abstract

We consider the K_4 -free process. In this process, the edges of the complete n -vertex graph are traversed in a uniformly random order, and each traversed edge is added to an initially empty evolving graph, unless the addition of the edge creates a copy of K_4 . Let $M(n)$ denote the graph that is produced by that process. We prove that a.a.s., the number of edges in $M(n)$ is $O(n^{8/5}(\ln n)^{1/5})$. This matches, up to a constant factor, a lower bound of Bohman. As a by-product, we prove the following Ramsey-type result: for every n there exists a K_4 -free n -vertex graph, in which the largest set of vertices that doesn't span a triangle has size $O(n^{3/5}(\ln n)^{1/5})$. This improves, by a factor of $(\ln n)^{3/10}$, an upper bound of Krivelevich.

1 Introduction

The K_4 -free process is a random greedy process that generates a K_4 -free graph. In this process, the edges in $\binom{[n]}{2}$, where $[n] := \{1, 2, \dots, n\}$, are traversed in a uniformly random order, and each traversed edge is added to an evolving graph, which is initially empty, unless the addition of the edge creates a copy of K_4 . Denote by $M(n)$ the (maximal) K_4 -free graph that is produced by that process. Say that an event holds asymptotically almost surely (a.a.s.) if the probability of that event goes to 1 as $n \rightarrow \infty$. Throughout the paper we assume that $n \rightarrow \infty$, and any asymptotic notation is used under this, and only this, assumption. The next theorem is our main result.

Theorem 1.1. *A.a.s., the number of edges in $M(n)$ is $O(n^{8/5}(\ln n)^{1/5})$.*

The first to study the K_4 -free process were Bollobás and Riordan [5], who showed that a.a.s., the number of edges in $M(n)$ is lower bounded by $\Omega(n^{8/5})$ and upper bounded by $O(n^{8/5} \ln n)$. Improved results were provided by Osthus and Taraz [15], who showed that a.a.s., the number of edges in $M(n)$ is upper bounded by $O(n^{8/5}(\ln n)^{1/2})$, and by Bohman [2], who showed that a.a.s., the number of edges in $M(n)$ is lower bounded by $\Omega(n^{8/5}(\ln n)^{1/5})$. Our main result shows that Bohman's lower bound is optimal up to the hidden constant factor.

The K_4 -free process is an instance of the more general H -free process, where instead of forbidding the appearance of a copy of K_4 in the evolving graph, one forbids the appearance of some fixed graph H . We will not discuss the H -free process here (we refer the reader to [3]), but we will say that the analysis of that process proves to be very useful in studying certain problems in extremal combinatorics. For example, the analysis of the H -free process in [2, 3] yielded the currently best

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known lower bounds for the off-diagonal Ramsey numbers $R(k, n)$, for every fixed $k \geq 4$. Our analysis of the K_4 -free process yields a new result regarding another Ramsey-type problem. We briefly discuss that problem now.

Let $2 \leq k < l \leq n$ be integers. For a graph G , let $f_k(G)$ be the maximum size of a subset of the vertices of G that spans no copy of K_k . Let $f_{k,l}(n)$ be defined to be $\min_G f_k(G)$, where the minimum is taken over all K_l -free n -vertex graphs G . For various choices of the parameters, the function $f_{k,l}(n)$ was studied in [1, 4, 6, 7, 13, 14, 19]. For the special case where $k = 3$ and $l = 4$, Krivelevich [13, 14] showed that $f_{3,4}(n) = \Omega((n \ln \ln n)^{1/2})$ and $f_{3,4}(n) = O(n^{3/5}(\ln n)^{1/2})$. The proof of Theorem 1.1, as we argue below, gives as a by-product the following result, which improves Krivelevich's upper bound by a factor of $(\ln n)^{3/10}$.

Theorem 1.2. $f_{3,4}(n) = O(n^{3/5}(\ln n)^{1/5})$.

The proof of Theorem 1.1 is based on analysing the following iterative process, which simulates the early stages of the K_4 -free process. This iterative process will be referred to throughout the paper simply as the process. Let $\varepsilon_1, \varepsilon_2$ and ε_3 be positive constants such that ε_1 is sufficiently small with respect to $\varepsilon_2 := 10^4 \varepsilon_3^3$, and ε_3 is sufficiently small. Let $I := \lfloor n^{\varepsilon_1 + \varepsilon_2} \rfloor$. Let M_0 and $Traversed_0$ be two empty graphs. Let $0 \leq i < I$ be an integer and suppose we have already defined M_i and $Traversed_i$. We define M_{i+1} and $Traversed_{i+1}$ as follows. Let $NotTraversed_i := \binom{[n]}{2} \setminus Traversed_i$. Let $BIGBite_{i+1}$ be a random set of edges constructed by taking every edge in $NotTraversed_i$ independently with probability $n^{\varepsilon_3 - 2/5}$. Let $BigBite_{i+1}$ be a random set of edges constructed by taking every edge in $BIGBite_{i+1}$ independently with probability $n^{\varepsilon_2 - \varepsilon_3}$. Let $Bite_{i+1}$ be a random set of edges constructed by taking every edge in $BigBite_{i+1}$ independently with probability $n^{-\varepsilon_1 - \varepsilon_2} / (1 - in^{-\varepsilon_1 - \varepsilon_2})$.¹ Assign each edge in $Bite_{i+1}$ a uniformly random birthtime in the unit interval and order the edges in $Bite_{i+1}$ by increasing birthtimes; traverse these edges according to that order and add each traversed edge to M_i , unless the addition of the edge creates a copy of K_4 . Let M_{i+1} be the graph thus constructed. Finally, let $Traversed_{i+1} := Traversed_i \cup Bite_{i+1}$ be the graph which is the set of edges that were already traversed.

Let $s := Cn^{3/5}(\ln n)^{1/5}$ be an integer, where $C = C(\varepsilon_1)$ is a sufficiently large constant. Observe that in order to prove Theorem 1.1, it is enough to prove the following theorem, which also easily implies Theorem 1.2 (as the existence of a K_4 -free n -vertex graph in which every set of s vertices spans a triangle implies $f_{3,4}(n) < s$).

Theorem 1.3. *A.a.s, every set of s vertices of M_I spans a triangle.*

The rest of the paper is devoted for the proof of Theorem 1.3. The argument that underlies the proof of Theorem 1.3 is an extension of the branching process argument of Spencer [20], which was used to give a limited, though non-trivial, analysis of the triangle-free process. We remark that our argument is different than the one used by Bohman [2] to analyze the K_4 -free process, an argument which is based on the differential equations method. Still, we note that there are similarities between the two arguments and so, wherever possible, we will reuse some of Bohman's results, instead of reproving them.

¹Equivalently, we could have defined $Bite_{i+1}$ to be a random set of edges constructed by taking every edge in $NotTraversed_i$ independently with probability $n^{-\varepsilon_1 - 2/5} / (1 - in^{-\varepsilon_1 - \varepsilon_2})$. The intermediate sets $BigBite_{i+1}$ and $BIGBite_{i+1}$ are introduced for technical reasons.

The paper is organized as follows. In Section 2 we state several probabilistic tools that we use throughout the paper. In Section 3 we give some basic definitions and state our main lemma. In Section 4 we study a certain branching process and a certain event – the event of survival – and in Section 5 we relate that event to the process. In Section 6 we prove several supporting lemmas that will be used in the proof of our main lemma. In Section 7 we use the results of the preceding sections to prove our main lemma. In Section 8 we use our main lemma to prove Theorem 1.3.

2 Probabilistic tools

Here we state several probabilistic tools that we use throughout the paper. We start with two deviation inequalities for random variables that count small subgraphs in the binomial random graph $G(n, p)$. (As usual, the binomial random graph $G(n, p)$ is the graph obtained by taking every edge in $\binom{[n]}{2}$ independently with probability p .) Our setting is as follows. For some index set L , let $\{G_l : l \in L\}$ be a family (potentially a multiset) of subgraphs of $\binom{[n]}{2}$, each of size $K = O(1)$. Let W count the number of indices $l \in L$ such that $G_l \subseteq G(n, p)$. In other words, let $W := \sum_{l \in L} \mathbf{1}[G_l \subseteq G(n, p)]$, where $\mathbf{1}[\mathcal{E}]$ is the indicator function of the event \mathcal{E} .

The first deviation inequality follows directly from a more general result of Vu [21, Corollary 4.4]. For $G \subseteq \binom{[n]}{2}$, let L_G be the set of all $l \in L$ such that $G \subseteq G_l$. Let $W_G := \sum_{l \in L_G} \mathbf{1}[G_l \setminus G \subseteq G(n, p)]$. For an integer $0 \leq k \leq K$, let $\mathbb{E}_k(W) := \max_{G: |G| \geq k} \mathbb{E}(W_G)$.

Theorem 2.1. *Let $\mathfrak{E}_0 > \mathfrak{E}_1 > \dots > \mathfrak{E}_K$ and λ be positive numbers such that $\mathfrak{E}_k \geq \mathbb{E}_k(W)$ for all $0 \leq k \leq K$, and $\mathfrak{E}_k/\mathfrak{E}_{k+1} \geq \lambda + 8k \ln n$ for all $0 \leq k \leq K-1$. Then for some positive constants c_1 and c_2 that depend only on K ,*

$$\Pr(|W - \mathbb{E}(W)| \geq c_1 \sqrt{\lambda \mathfrak{E}_0 \mathfrak{E}_1}) \leq c_2 \exp(-\lambda/4).$$

The second deviation inequality follows directly from the more general Janson's inequality [9] (see also [12, Theorem 2.14]). Let $\Delta := \sum_{\{l, l'\}} \mathbb{E}(\mathbf{1}[G_l, G_{l'} \subseteq G(n, p)])$, where the sum ranges over all sets $\{l, l'\} \subseteq L$ such that $l \neq l'$ and $G_l \cap G_{l'} \neq \emptyset$.

Theorem 2.2. *For some absolute positive constant c_3 , for all $0 \leq \lambda \leq \mathbb{E}(W)$,*

$$\Pr(W \leq \mathbb{E}(W) - \lambda) \leq \exp\left(-\frac{c_3 \lambda^2}{\mathbb{E}(W) + \Delta}\right).$$

Another result that we use applies to the same setting as above, and follows directly from a result of Janson and Ruciński [10, 11].

Theorem 2.3. *Assume that $\{G_l : l \in L\}$ is a set (that is, not a multiset). For every pair of positive real numbers r and λ , with probability at least $1 - \exp\left(-\frac{r\lambda}{K(\mathbb{E}(W) + \lambda)}\right)$, there is a set $E_0 \subseteq G(n, p)$ of size at most r , such that $G(n, p) \setminus E_0$ contains fewer than $\mathbb{E}(W) + \lambda$ members of $\{G_l : l \in L\}$.*

We end this section by stating McDiarmid's inequality [16]. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be independent random variables with α_i taking values in a set A_i . Let $\varphi : \prod_{i=1}^m A_i \rightarrow \mathbb{R}$ satisfy the following

Lipschitz condition: if two vectors $\alpha, \alpha' \in \prod_{i=1}^m A_i$ differ only in the i th coordinate, then $|\varphi(\alpha) - \varphi(\alpha')| \leq a_i$. Redefine $W := \varphi(\alpha_1, \alpha_2, \dots, \alpha_m)$. McDiarmid's inequality states that for any $\lambda \geq 0$,

$$\Pr(|W - \mathbb{E}(W)| \geq \lambda) \leq 2 \exp \left(- \frac{2\lambda^2}{\sum_{i=1}^m a_i^2} \right).$$

3 Main lemma

The purpose of this section is to state our main lemma. We start with some basic definitions. Let $\Phi(x)$ be a function over the Reals, whose derivative is $\phi(x) := \exp(-0.5\Phi(x)^5)$, and which satisfies $\Phi(0) = 0$. For integers i, j and t , let

$$\begin{aligned} x_{i,j} &:= \binom{n}{2} \binom{5}{j} \left(\frac{\Phi(in^{-\varepsilon_1})}{n^{2/5}} \right)^{5-j} \phi(in^{-\varepsilon_1})^j \quad \text{and} \\ y_{i,j,t} &:= t \binom{3}{j} \left(\frac{\Phi(in^{-\varepsilon_1})}{n^{2/5}} \right)^{3-j} \phi(in^{-\varepsilon_1})^j. \end{aligned}$$

In addition, let

$$\begin{aligned} z_{i,j} &:= n^{(\varepsilon_2 - 2/5)j} x_{i,j}, \quad \gamma_i := 2 \prod_{1 \leq j \leq 5} (1 - 2n^{-\varepsilon_1 j - \varepsilon_2 j})^{-6000z_{i,j}} - 2 \quad \text{and} \\ \Gamma_i &:= \begin{cases} n^{-\varepsilon_1} & \text{if } i = 0, \\ \Gamma_{i-1}(1 + \gamma_{i-1}) & \text{if } i \geq 1. \end{cases} \end{aligned}$$

We remark that for every $0 \leq i \leq I$, $\Gamma_i \rightarrow 0$ as $n \rightarrow \infty$. This is implied by the next lemma (whose proof is given below), which also gives other useful facts that will be used throughout the paper.

Lemma 3.1. *For an integer $0 \leq i \leq I$,*

- (i) $in^{-\varepsilon_1} \rightarrow \infty \implies \phi(in^{-\varepsilon_1}) = \Theta(1/(in^{-\varepsilon_1}(\ln in^{-\varepsilon_1})^{4/5}))$ and $\Phi(in^{-\varepsilon_1}) = \Theta((\ln in^{-\varepsilon_1})^{1/5})$;
- (ii) $n^{-\Theta(\varepsilon_1)} \leq \phi(in^{-\varepsilon_1}) \leq 1$ and $i \geq 1 \implies 0.9n^{-\varepsilon_1} \leq \Phi(in^{-\varepsilon_1}) = O((\ln n)^{1/5})$;
- (iii) $\gamma_i = n^{-\Theta(\varepsilon_1)}$ and $\Gamma_i = n^{-\Theta(\varepsilon_1)}$.

For an integer $0 \leq i < I$, let O_i be the set of edges $f \in \text{NotTraversed}_i$ such that $M_i \cup \{f\}$ is K_4 -free. Furthermore, for an integer $0 \leq j \leq 5$ and an edge $f \in \text{NotTraversed}_i$, let $X_{i,j}(f)$ be the set of all graphs $G \subseteq M_i \cup \text{NotTraversed}_i$ such that $|G| = 5$, $G \cup \{f\}$ is isomorphic to K_4 , $|M_i \cap G| = 5 - j$, $|\text{NotTraversed}_i \cap G| = j$, and $M_i \cup \{g\}$ is K_4 -free for all $g \in G$ (or equivalently, $g \in O_i$ for all $g \in \text{NotTraversed}_i \cap G$). Lastly, let O'_i be the set of all $f \in O_i$ such that $f \in \text{BIGBite}_{i+1}$, and let $X'_{i,j}(f)$ be the set of all $G \in X_{i,j}(f)$ such that $G \subseteq M_i \cup \text{BIGBite}_{i+1}$.

For integers $0 \leq i < I$ and $1 \leq j \leq 3$, and for a set of T of triangles in $\binom{[n]}{3}$, let $Y_{i,j}(T)$ be the set of all triangles $G \in T$ such that $|M_i \cap G| = 3 - j$, $|\text{NotTraversed}_i \cap G| = j$, and $M_i \cup G$ is K_4 -free. Furthermore, for an integer $1 \leq k < j$, let $Y_{i,j,k}(T)$ be the set of all triples (G_1, G_2, G_3) such that for some $G \in Y_{i,j}(T)$, $G_1 = M_i \cap G$, $G_2 \subset \text{NotTraversed}_i \cap G$ with $|G_2| = k$, and $G_3 = G \setminus (G_1 \cup G_2)$. (Note that $|Y_{i,j,k}(T)| = \binom{j}{k} |Y_{i,j}(T)|$.) Lastly, let $Y'_{i,j,k}(T)$ be the set of all triples $(G_1, G_2, G_3) \in Y_{i,j,k}(T)$ such that $G_2 \subseteq \text{BIGBite}_{i+1}$.

For an integer $0 \leq i < I$, for a set $R \subseteq [n]$, and for a set T of triangles in $\binom{R}{2}$, let $Z_i(R, T)$ be the set of all triangles $G \in T$ such that $|M_i \cap G| = 2$, $|NotTraversed_i \cap G| = 1$, and letting g denote the edge in $NotTraversed_i \cap G$, there exists $G_0 \in X_{i,0}(g)$ such that G_0 shares at least three vertices with R .

For a set $S \subseteq [n]$ with $s - o(s) \leq |S| \leq s$, let $Pairs(S)$ be the set of all pairs (R, T) such that $R \subseteq S$ has size $s - o(s) \leq |R| \leq |S|$, and for some partition of R to three sets of size $\Omega(s)$ each, T is the set of all triangles in $\binom{R}{2}$ that have each exactly one vertex in each part of the partition.

Next, we define a few events. These events will be used to track the random variables defined above, as they evolve throughout the process, as well as to track some other properties of the process. In what follows, and throughout the paper, an expression that contains a symbol \pm is a shorthand for the interval $[\eta_-, \eta_+]$, where η_- is obtained by replacing in the expression every \pm with $-$, and η_+ is obtained by replacing in the expression every \pm with $+$.

- Let \mathcal{A}_i be the event that the following properties hold:

$$(A1) \quad |M_i| \in 0.5n^{8/5}\Phi(in^{-\varepsilon_1})(1 \pm 100\Gamma_i);$$

$$(A2) \quad |O_i| \in 0.5n^2\phi(in^{-\varepsilon_1})(1 \pm 100\Gamma_i);$$

$$(A3) \quad |X_{i,j}(f)| \in x_{i,j}(1 \pm 1000\Gamma_i) \text{ for all } 1 \leq j \leq 5 \text{ and all } f \in NotTraversed_i.$$

- Let \mathcal{B}_i be the event that for every set $S \subseteq [n]$ of s vertices, there is a set $S_i \subseteq S$, such that the following properties hold:

$$(B1) \quad S_i \text{ has size at least } s(1 - in^{-0.01});$$

$$(B2) \quad Traversed_i \cap \binom{S_i}{2} \text{ has maximum degree at most } n^{1.1/5};$$

$$(B3) \quad |Y_{i,j}(T)| \geq y_{i,j,t}(1 - 100\Gamma_i) - 0.5j(3-j)(2-j)|Z_i(R, T)| \text{ for all } 1 \leq j \leq 3, \text{ and for every pair } (R, T) \in Pairs(S_i) \text{ with } |T| = t. \text{ (Note that the coefficient of } |Z_i(R, T)| \text{ is equal to 0 if } 2 \leq j \leq 3 \text{ and is equal to 1 if } j = 1.)$$

- Let \mathcal{C}_i be the event that the following properties hold:

$$(C1) \quad \text{the number of edges in } Traversed_i \cup BIGBite_{i+1} \text{ is at most } n^{8/5+10\varepsilon_3};$$

$$(C2) \quad \text{for every set } S \subseteq [n] \text{ of } s \text{ vertices, there are at most } n^{4/5+10\varepsilon_3} \text{ edges in } (Traversed_i \cup BIGBite_{i+1}) \cap \binom{S}{2};$$

$$(C3) \quad \text{for every two vertices } v_1, v_2 \in [n], \text{ there are at most } n^{1/5+10\varepsilon_3} \text{ other vertices in } [n] \text{ that are adjacent in } Traversed_i \cup BIGBite_{i+1} \text{ simultaneously to } v_1 \text{ and } v_2; \text{ moreover, for every three vertices } v_1, v_2, v_3 \in [n], \text{ there are at most } (\ln n)^{O(1)} \text{ other vertices in } [n] \text{ that are adjacent in } Traversed_i \cup BIGBite_{i+1} \text{ simultaneously to } v_1, v_2 \text{ and } v_3;$$

$$(C4) \quad \text{for every edge } f, \text{ there are at most } (\ln n)^{O(1)} \text{ pairs } (G, v) \text{ such that } G \in X_{0,5}(f), G \subseteq Traversed_i \cup BIGBite_{i+1}, \text{ and } v \text{ is a vertex outside of the vertex set (of size four) of } G \text{ which is adjacent in } Traversed_i \cup BIGBite_{i+1} \text{ to at least three vertices of } G;$$

$$(C5) \quad \text{for every edge } f \in Traversed_i \cup BIGBite_{i+1}, \text{ there are at most } n^{2/5+10\varepsilon_3} \text{ copies of } K_4^- \text{ (which is a } K_4 \text{ without an edge) in } Traversed_i \cup BIGBite_{i+1} \text{ which contain } f;$$

- (C6) for every set $S \subseteq [n]$ of s vertices, there is a set of at most $n^{3/5+10\epsilon_3}$ edges, the removal of which from $Traversed_i \cup BIGBite_{i+1}$ leaves at most $n^{4/5+10\epsilon_3}$ 4-cycles in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{S}{2}$;
- (C7) for every set $R \subseteq [n]$ of r vertices, where $s - o(s) \leq r \leq s$, the following holds for every $M \subseteq Traversed_i \cup BIGBite_{i+1}$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$. First, there are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq M$, which shares all four vertices with R . Second, there is a set $R_0 \subseteq [n] \setminus R$ of at most $n^{0.99/5}$ vertices, such that there are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq M$, which shares exactly three vertices with R and one vertex with $[n] \setminus (R \cup R_0)$;
- (C8) for every set $R \subseteq [n]$ of r vertices, where $s - o(s) \leq r \leq s$, and for every set E of $O(n^{1/2})$ edges in $\binom{[n]}{2} \setminus \binom{R}{2}$, the following holds, assuming the maximum degree in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{R}{2}$ is at most $n^{1.1/5}$. There are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ for which there exists a path of length two in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{R}{2}$ that completes g to a triangle, and a graph $G \in X_{0,5}(g)$, with $G \subseteq (Traversed_i \cup BIGBite_{i+1}) \setminus \binom{R}{2}$ and $G \cap E \neq \emptyset$.

- Let \mathcal{D}_i be the event that the following properties hold:

- (D1) $|O'_i| \in 0.5n^{8/5+\epsilon_3}\phi(in^{-\epsilon_1})(1 \pm (100\Gamma_i + \Gamma_i\gamma_i))$;
- (D2) $|X'_{i,j}(f)| \in n^{(\epsilon_3-2/5)j}x_{i,j}(1 \pm 2000\Gamma_i)$ for all $1 \leq j \leq 5$ and all $f \in NotTraversed_i$. In particular, $|X'_{i,j}(f)| \leq n^{\epsilon_3j+o(1)} \leq n^{10\epsilon_3}$ for all $1 \leq j \leq 5$ and all $f \in NotTraversed_i$;
- (D3) for every set $S \subseteq [n]$ of s vertices, assuming \mathcal{B}_i holds, letting $S_i \subseteq S$ be the set that is guaranteed to exist by \mathcal{B}_i ,

$$|Y'_{i,j,k}(T)| \geq n^{(\epsilon_3-2/5)k} \binom{j}{k} y_{i,j,t} (1 - 100\Gamma_i - \Gamma_i\gamma_i)$$

for all $1 \leq k < j \leq 3$, and for every pair $(R, T) \in Pairs(S_i)$ with $|T| = t$.

Finally, we state our main lemma.

Lemma 3.2. For $0 \leq i < I$, $\Pr(\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq 1 - in^{-0.1} - n^{-\omega(1)}$.

3.1 Proof of Lemma 3.1

A standard analysis of the separable differential equation $\phi(x) = \exp(-0.5\Phi(x)^5)$ with the initial condition $\Phi(0) = 0$ gives the estimates in the first two items. It also shows that $\Phi(in^{-\epsilon_1})$ is monotonically increasing with i . We prove the validity of the third item.

From the definition of $z_{i,j}$ it follows that

$$z_{i,j} = \Theta(n^{\epsilon_2j} \cdot \Phi(in^{-\epsilon_1})^{5-j} \cdot \phi(in^{-\epsilon_1})^j).$$

Plugging this into the definition of γ_i , we get that for all $0 \leq i \leq I$,

$$\gamma_i = 2 \exp \left(\Theta(1) \cdot \sum_{1 \leq j \leq 5} n^{-\epsilon_1j} \cdot \Phi(in^{-\epsilon_1})^{5-j} \cdot \phi(in^{-\epsilon_1})^j \right) - 2.$$

This, together with the second item, implies that for all $0 \leq i \leq I$,

$$n^{-\Theta(\varepsilon_1)} \leq \gamma_i = O(n^{-\varepsilon_1} + n^{-\varepsilon_1} \cdot \Phi(in^{-\varepsilon_1})^4 \cdot \phi(in^{-\varepsilon_1})), \quad (1)$$

and if in addition $in^{-\varepsilon_1} \rightarrow \infty$, then

$$\gamma_i = O(n^{-\varepsilon_1} \cdot \Phi(in^{-\varepsilon_1})^4 \cdot \phi(in^{-\varepsilon_1})). \quad (2)$$

In particular, from (1) and the second item, we get that $\gamma_i = n^{-\Theta(\varepsilon_1)}$ for all $0 \leq i \leq I$.

To show that $\Gamma_i = n^{-\Theta(\varepsilon_1)}$ for all $0 \leq i \leq I$, it suffices to show that $\Gamma_I \leq n^{-\Theta(\varepsilon_1)}$. For $0 \leq i \leq I$, by (1) and by the second item,

$$\gamma_i = O(n^{-\varepsilon_1} + n^{-\varepsilon_1} \cdot \Phi(in^{-\varepsilon_1})^4).$$

Therefore, letting $i_0 := \lfloor n^{\varepsilon_1} \ln \ln n \rfloor$, for $0 \leq i \leq i_0$, by the first item and the monotonicity of $\Phi(in^{-\varepsilon_1})$, $\gamma_i = O(n^{-\varepsilon_1} \ln \ln n)$. Thus,

$$\Gamma_{i_0} = n^{-\varepsilon_1} \cdot \prod_{0 \leq i < i_0} (1 + \gamma_i) = n^{-\varepsilon_1 + o(1)}.$$

By (2) and the first item, if $i_0 \leq i \leq I$, then

$$\gamma_i = O\left(\frac{1}{i}\right).$$

Therefore, recalling that $I = \lfloor n^{\varepsilon_1 + \varepsilon_1^2} \rfloor$,

$$\Gamma_I = \Gamma_{i_0} \cdot \prod_{i_0 \leq i < I} (1 + \gamma_i) = n^{-\varepsilon_1 + o(1)} \cdot \exp\left(\sum_{i_0 \leq i < I} O\left(\frac{1}{i}\right)\right) = n^{-\varepsilon_1 + O(\varepsilon_1^2)}.$$

4 Survival

The purpose of this section is to define and analyze a certain event, an event which in the next section will be related to the process, and which later on will be used in the analysis of the process. Fix for the rest of the section an integer $0 \leq i < I$ and a sufficiently large integer c which we will assume to be constant, independent of n . Let \mathfrak{T}_1 be a rooted, finite tree with the following three properties: first, each leaf in the tree is at distance $2c$ from the root; second, every non-leaf node at even distance from the root has 5 sets of children, where the j th set has size in $z_{i,j}(1 \pm 3000\Gamma_i)$ and consists of sets of size j ; third, every node at odd distance from the root which is a set of size j has exactly j children. (We remark that we don't use the fact that a node at odd distance from the root of \mathfrak{T}_1 is a set of size j , except to indicate that such a node has j children.) Assign each node ν at even distance from the root of \mathfrak{T}_1 a uniformly random birthtime $\beta(\nu)$ in the unit interval. Let ν_0 be a node at even distance from the root of \mathfrak{T}_1 . Define the event that ν_0 survives as follows. If ν_0 is a leaf then ν_0 survives by definition; otherwise, ν_0 survives if and only if for every child ν_1 of ν_0 , the following holds: if for every child ν_2 of ν_1 we have $\beta(\nu_2) < \beta(\nu_0)$ then ν_1 has a child that does not survive.

Let $\nu_{\mathfrak{T}_1}$ denote the root of \mathfrak{T}_1 . Let $p_{\mathfrak{T}_1}(x)$ be the probability that $\nu_{\mathfrak{T}_1}$ survives, under the assumption that its birthtime is equal to $xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$; in other words, we define β as above, only that now we further set the birthtime of $\nu_{\mathfrak{T}_1}$ to be $xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$, and ask for the probability $p_{\mathfrak{T}_1}(x)$ that $\nu_{\mathfrak{T}_1}$ survives given that setup. Let $P_{\mathfrak{T}_1}(x) := x \Pr(\nu_{\mathfrak{T}_1} \text{ survives} \mid \beta(\nu_{\mathfrak{T}_1}) \leq xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2}))$ if $x > 0$ and $P_{\mathfrak{T}_1}(x) := 0$ if $x = 0$. The main result of this section follows.

Lemma 4.1. $p_{\mathfrak{T}_1}(n^{-\varepsilon_1}) \in \frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})}(1 \pm 20\Gamma_i\gamma_i)$; $P_{\mathfrak{T}_1}(n^{-\varepsilon_1}) \in \frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})}(1 \pm 20\Gamma_i\gamma_i)$.

4.1 A related event

Let \mathfrak{T}_2 be a rooted tree with the following three properties: first, each leaf in the tree is at even distance from the root; second, every node at even distance less than $2c$ from the root has 5 sets of children, where the j th set has size in $z_{i,j}(1 \pm 3000\Gamma_i)$ and consists of sets of size j ; third, every node at odd distance less than $2c$ from the root which is a set of size j has exactly j children. (Note that we make no assumptions as for the number of children of nodes at distance at least $2c$ from the root of \mathfrak{T}_2 . In particular, it is possible for \mathfrak{T}_2 to have an infinite path.) Extend β by assigning each node ν at even distance from the root of \mathfrak{T}_2 a uniformly random birthtime $\beta(\nu)$ in the unit interval. Define the event that a node at even distance from the root of \mathfrak{T}_2 survives exactly as it was defined for such a node in \mathfrak{T}_1 . Let $\nu_{\mathfrak{T}_2}$ denote the root of \mathfrak{T}_2 . Let $p_{\mathfrak{T}_2}(x)$ be the probability that $\nu_{\mathfrak{T}_2}$ survives, under the assumption that its birthtime is equal to $xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$. Let $P_{\mathfrak{T}_2}(x) := x \Pr(\nu_{\mathfrak{T}_2} \text{ survives} \mid \beta(\nu_{\mathfrak{T}_2}) \leq xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2}))$ if $x > 0$ and $P_{\mathfrak{T}_2}(x) := 0$ if $x = 0$. The purpose of this subsection is to prove the following lemma, which relates the event that the root of \mathfrak{T}_1 survives to the event that the root of \mathfrak{T}_2 survives.

Lemma 4.2. $p_{\mathfrak{T}_1}(n^{-\varepsilon_1}) \in p_{\mathfrak{T}_2}(n^{-\varepsilon_1})(1 \pm 5\Gamma_i\gamma_i)$ and $P_{\mathfrak{T}_1}(n^{-\varepsilon_1}) \in P_{\mathfrak{T}_2}(n^{-\varepsilon_1})(1 \pm 5\Gamma_i\gamma_i)$.

Let \mathfrak{T}_3 be obtained by removing from \mathfrak{T}_2 every subtree that is rooted at a node at distance larger than $2c$ from $\nu_{\mathfrak{T}_2}$. Observe that \mathfrak{T}_3 satisfies the exact same three properties that \mathfrak{T}_1 satisfies. Define the event that a node at even distance from the root of \mathfrak{T}_3 survives exactly as it was defined for such a node in \mathfrak{T}_1 . (Note that a node in \mathfrak{T}_3 is also a node in \mathfrak{T}_2 , but the event that such a node survives with \mathfrak{T}_2 being the underlying tree is not necessarily the same as the event that such a node survives with \mathfrak{T}_3 being the underlying tree. Below, when stating that a node in \mathfrak{T}_3 survives, the exact tree that underlies this event should be understood from the context. For example, when stating that $\nu_{\mathfrak{T}_2}$ survives we refer to the event that the root of \mathfrak{T}_2 survives and not to the event that the root of \mathfrak{T}_3 survives.) Let $\nu_{\mathfrak{T}_3}$ denote the root of \mathfrak{T}_3 . Let $p_{\mathfrak{T}_3}(x)$ be the probability that $\nu_{\mathfrak{T}_3}$ survives under the assumption that its birthtime is equal to $xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$. Let $P_{\mathfrak{T}_3}(x) := x \Pr(\nu_{\mathfrak{T}_3} \text{ survives} \mid \beta(\nu_{\mathfrak{T}_3}) \leq xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2}))$ if $x > 0$ and $P_{\mathfrak{T}_3}(x) := 0$ if $x = 0$. The next two lemmas imply Lemma 4.2.

Lemma 4.3. For $0 \leq x \leq n^{-\varepsilon_1}$, $p_{\mathfrak{T}_3}(x) \in p_{\mathfrak{T}_2}(x)(1 \pm \Gamma_i\gamma_i)$.

Proof. We prove the lemma under the assumption that c is odd. The proof for the case where c is even is similar and will be omitted.

Fix $0 \leq x \leq n^{-\varepsilon_1}$ and assume that $\beta(\nu_{\mathfrak{T}_2}) = \beta(\nu_{\mathfrak{T}_3}) = xn^{-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2}) < 2xn^{-\varepsilon_2}$. Since c is odd, the event that $\nu_{\mathfrak{T}_3}$ survives implies the event that $\nu_{\mathfrak{T}_2}$ survives. Hence $p_{\mathfrak{T}_3}(x) \leq p_{\mathfrak{T}_2}(x)$.

Below we show that $p_{\mathfrak{T}_3}(x) \geq p_{\mathfrak{T}_2}(x) - n^{-\Theta(\varepsilon_1 c)}$. Note that $p_{\mathfrak{T}_2}(x) = \Omega(1)$. (Indeed, a sufficient condition for the event that $\nu_{\mathfrak{T}_2}$ survives is that for every child ν_1 of $\nu_{\mathfrak{T}_2}$, there is a child ν_2 of ν_1 with $\beta(\nu_2) > \beta(\nu_{\mathfrak{T}_2})$. Given the above assumption on $\beta(\nu_{\mathfrak{T}_2})$ and the properties of \mathfrak{T}_2 , this event occurs with probability $\Omega(1)$.) Therefore, $p_{\mathfrak{T}_3}(x) \geq p_{\mathfrak{T}_2}(x)(1 - n^{-\Theta(\varepsilon_1 c)})$. Since c is sufficiently large, it follows from Lemma 3.1 that $\Gamma_i \gamma_i \geq n^{-\Theta(\varepsilon_1 c)}$. This gives the lemma.

Say that a non-root node ν at even distance from the root of \mathfrak{T}_3 is relevant, if the following two properties hold: first, the grandparent of ν has a larger birthtime than the birthtime of ν and the birthtimes of ν 's siblings (if there are any); second the grandparent of ν is either relevant or the root. Observe that if the root of \mathfrak{T}_2 survives then either the root of \mathfrak{T}_3 survives, or else, there is a relevant leaf in \mathfrak{T}_3 . Thus, it remains to show that the expected number of relevant leaves in \mathfrak{T}_3 is at most $n^{-\Theta(\varepsilon_1 c)}$.

Say that a leaf ν in \mathfrak{T}_3 is an $(a_1, a_2, a_3, a_4, a_5)$ -type, if the path leading from the root to ν contains exactly a_j nodes at odd distance from the root which are sets of size j . Consider a path $(\nu_{\mathfrak{T}_3}, \nu_1, \nu_2, \dots, \nu_{2c})$ from the root to a leaf, where the leaf ν_{2c} is an $(a_1, a_2, a_3, a_4, a_5)$ -type. Given such a path, let N be the set of nodes which is the union of $\{\nu_{2b} : 1 \leq b \leq c\}$ together with $\{\nu : \nu \text{ is a sibling of some } \nu_{2b} \text{ for some } 1 \leq b \leq c\}$. Note that $|N| = \sum_{1 \leq j \leq 5} j a_j = \Theta(c)$. Now, if ν_{2c} is relevant, then the birthtime of every node in N is less than $2x n^{-\varepsilon_2}$. This event occurs with probability $(2x n^{-\varepsilon_2})^{|N|}$. The number of $(a_1, a_2, a_3, a_4, a_5)$ -type leaves in \mathfrak{T}_3 is at most $4^{5c} \prod_{1 \leq j \leq 5} z_{i,j}^{a_j}$. Hence, the expected number of relevant $(a_1, a_2, a_3, a_4, a_5)$ -type leaves in \mathfrak{T}_3 is at most

$$(2x n^{-\varepsilon_2})^{|N|} \cdot 4^{5c} \prod_{1 \leq j \leq 5} z_{i,j}^{a_j} = 4^{5c} \cdot \prod_{1 \leq j \leq 5} (2x n^{-\varepsilon_2})^{j a_j} z_{i,j}^{a_j} \leq n^{-\Theta(\varepsilon_1 c)},$$

where the inequality follows since $z_{i,j} \leq n^{\varepsilon_2 j + o(1)}$ by Lemma 3.1 and since $x \leq n^{-\varepsilon_1}$. To complete the proof, note that if a leaf is an $(a_1, a_2, a_3, a_4, a_5)$ -type then the number of choices we have for $\{a_j : 1 \leq j \leq 5\}$ is at most $(c+1)^5$. A union bound argument now finishes the proof. \blacksquare

Lemma 4.4. For $0 \leq x \leq n^{-\varepsilon_1}$, $p_{\mathfrak{T}_1}(x) \in p_{\mathfrak{T}_3}(x)(1 \pm \Gamma_i \gamma_i)$.

Proof. For a node ν in a tree \mathfrak{T}_* , let $p_{\mathfrak{T}_*,\nu}(x)$ be the probability that ν survives under the assumption that $\beta(\nu) = x n^{-\varepsilon_2} / (1 - i n^{-\varepsilon_1 - \varepsilon_2})$ and furthermore, let $P_{\mathfrak{T}_*,\nu}(x) := x \Pr(\nu \text{ survives} | \beta(\nu) \leq x n^{-\varepsilon_2} / (1 - i n^{-\varepsilon_1 - \varepsilon_2}))$ if $x > 0$ and $P_{\mathfrak{T}_*,\nu}(x) := 0$ if $x = 0$. The following implies the lemma.

Claim 4.5. Let $0 \leq x \leq n^{-\varepsilon_1}$. Let $0 \leq b \leq c$ be an integer. If ν is a node at height $2b$ in \mathfrak{T}_1 and μ is a node at height $2b$ in \mathfrak{T}_3 , then $p_{\mathfrak{T}_1,\nu}(x) \in p_{\mathfrak{T}_3,\mu}(x)(1 \pm \Gamma_i \gamma_i)$.

The proof of the claim is by induction on b . For $b = 0$, both ν and μ are leaves and so the claim holds since by definition $p_{\mathfrak{T}_1,\nu}(x) = p_{\mathfrak{T}_3,\mu}(x) = 1$ for all $0 \leq x \leq n^{-\varepsilon_1}$. Let $1 \leq b \leq c$ be an integer and assume the claim holds for $b-1$, for all $0 \leq x \leq n^{-\varepsilon_1}$. Note that by the induction hypothesis, if ν' is a node at height $2(b-1)$ in \mathfrak{T}_1 and μ' is a node at height $2(b-1)$ in \mathfrak{T}_3 then for all $0 \leq x \leq n^{-\varepsilon_1}$,

$$P_{\mathfrak{T}_1,\nu'}(x) \in P_{\mathfrak{T}_3,\mu'}(x)(1 \pm \Gamma_i \gamma_i).$$

Fix $0 \leq x \leq n^{-\varepsilon_1}$, a node ν at height $2b$ in \mathfrak{T}_1 and a node μ at height $2b$ in \mathfrak{T}_3 . Recall that \mathfrak{T}_1 and \mathfrak{T}_3 satisfy the same properties, and so it is enough to prove that $p_{\mathfrak{T}_1,\nu} \leq p_{\mathfrak{T}_3,\mu}(1 +$

$\Gamma_i \gamma_i$). Let $Children(\cdot)$ denote the set of children of a given node in either \mathfrak{T}_1 or \mathfrak{T}_3 . Let $N_j := \{Children(\nu_1) : \nu_1 \text{ is a child of } \nu \text{ which is a set of size } j\}$, and likewise, let $L_j := \{Children(\mu_1) : \mu_1 \text{ is a child of } \mu \text{ which is a set of size } j\}$. It is safe to assume that $z_{i,j}(1 - 3000\Gamma_i) \leq |N_j| \leq |L_j|$ for every j (since otherwise we can remove some of the subtrees that are rooted at some of the children of ν so that this assumption does hold; such an alteration will only increase the probability that ν survives). Let l_j be an injective function that associates each set in N_j with a unique set in L_j . For brevity, let $\zeta := 1/(1 - in^{-\varepsilon_1 - \varepsilon_2})$. We have

$$\begin{aligned} p_{\mathfrak{T}_1, \nu}(x) &= \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\nu' \in S} P_{\mathfrak{T}_1, \nu'}(x)\right) \\ &\leq \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\mu' \in l_j(S)} (P_{\mathfrak{T}_3, \mu'}(x)(1 - \Gamma_i \gamma_i))\right) \\ &\leq \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\mu' \in l_j(S)} P_{\mathfrak{T}_3, \mu'}(x)\right)^{1 - 2j\Gamma_i \gamma_i} = (*), \end{aligned}$$

where the first equality is by definition, the first inequality is by the induction hypothesis, and the second inequality follows from known exponential inequalities (i.e., the fact that for $a > 1$, $\exp(-1/(a-1)) \leq 1 - 1/a \leq \exp(-1/a)$), together with the fact that $\zeta^j n^{-\varepsilon_2 j} = o(\Gamma_i \gamma_i)$ (which follows from Lemma 3.1) and the fact that $P_{\mathfrak{T}_3, \mu'}(x) \leq x \leq n^{-\varepsilon_1}$ for all μ' . This upper bound on $P_{\mathfrak{T}_3, \mu'}(x)$, together with the fact that $|N_j| \leq 2z_{i,j}$ and with Lemma 3.1, gives

$$\begin{aligned} \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\mu' \in l_j(S)} P_{\mathfrak{T}_3, \mu'}(x)\right)^{-2j\Gamma_i \gamma_i} &\leq \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_1 j - \varepsilon_2 j}\right)^{-2j\Gamma_i \gamma_i} \\ &\leq \prod_{1 \leq j \leq 5} \left(1 - \zeta^j n^{-\varepsilon_1 j - \varepsilon_2 j}\right)^{-4j\Gamma_i \gamma_i z_{i,j}} \\ &\leq 1 + o(\Gamma_i \gamma_i). \end{aligned}$$

Moreover, letting $L'_j := L_j \setminus \{l_j(S) : S \in N_j\}$, we have

$$\begin{aligned} \prod_{1 \leq j \leq 5} \prod_{S \in N_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\mu' \in l_j(S)} P_{\mathfrak{T}_3, \mu'}(x)\right) &= p_{\mathfrak{T}_3, \mu}(x) \prod_{1 \leq j \leq 5} \prod_{S \in L'_j} \left(1 - \zeta^j n^{-\varepsilon_2 j} \prod_{\mu' \in S} P_{\mathfrak{T}_3, \mu'}(x)\right)^{-1} \\ &\leq p_{\mathfrak{T}_3, \mu}(x) \prod_{1 \leq j \leq 5} \left(1 - 2n^{-\varepsilon_1 j - \varepsilon_2 j}\right)^{-6000\Gamma_i z_{i,j}} \\ &\leq p_{\mathfrak{T}_3, \mu}(x)(1 + 0.5\Gamma_i \gamma_i), \end{aligned}$$

where the equality follows from the definition of $p_{\mathfrak{T}_3, \mu}(x)$, the first inequality follows since $\zeta^j \leq 2$, since $P_{\mathfrak{T}_3, \mu'}(x) \leq x \leq n^{-\varepsilon_1}$ for all μ' , and since $|L'_j| = |L_j| - |N_j| \leq 6000\Gamma_i z_{i,j}$, and the second inequality follows using the definition of γ_i . It follows that $(*) \leq p_{\mathfrak{T}_3, \mu}(x)(1 + \Gamma_i \gamma_i)$. \blacksquare

4.2 Proof of Lemma 4.1

Let \mathfrak{T}_4 be an infinite tree with the following two properties: first, each node at even distance from the root has $0.5n^{5\varepsilon_2}$ children; second, each node at odd distance from the root has 5 children. (Note that we implicitly assume that $0.5n^{5\varepsilon_2}$ is an integer. This is a safe assumption since we can always

choose ε_2 so that this assumption holds.) Extend β by assigning each node ν at even distance from the root of \mathfrak{T}_4 a uniformly random birthtime $\beta(\nu)$ in the unit interval. Define the event that a node at even distance from the root of \mathfrak{T}_4 survives exactly as it was defined for such a node in \mathfrak{T}_1 . Let $\nu_{\mathfrak{T}_4}$ denote the root of \mathfrak{T}_4 . Let $p_{\mathfrak{T}_4}(x)$ be the probability that $\nu_{\mathfrak{T}_4}$ survives under the assumption that its birthtime is equal to $xn^{-\varepsilon_2}$, at the limit as $n \rightarrow \infty$. It is not hard to see that $p_{\mathfrak{T}_4}(x)$ is continuous and bounded. Hence $p_{\mathfrak{T}_4}(x)$ is integrable. Let $P_{\mathfrak{T}_4}(x) := \int_0^x p_{\mathfrak{T}_4}(y)dy$. Note that for $0 < x \leq n^{\varepsilon_2}$, $P_{\mathfrak{T}_4}(x) = x \Pr(\nu_{\mathfrak{T}_4} \text{ survives} \mid \beta(\nu_{\mathfrak{T}_4}) \leq xn^{-\varepsilon_2})$.

Lemma 4.6. *For $0 \leq x \leq n^{\varepsilon_2}$, $p_{\mathfrak{T}_4}(x) = \phi(x)$ and $P_{\mathfrak{T}_4}(x) = \Phi(x)$.*

Proof. By definition, for every $0 \leq x \leq n^{\varepsilon_2}$,

$$p_{\mathfrak{T}_4}(x) = \lim_{n \rightarrow \infty} \left(1 - n^{-5\varepsilon_2} P_{\mathfrak{T}_4}(x)^5\right)^{0.5n^{5\varepsilon_2}} = \exp\left(-0.5P_{\mathfrak{T}_4}(x)^5\right). \quad (3)$$

(Indeed, if $0 < x \leq n^{\varepsilon_2}$ then the first equality above holds by definition of the event that the root of \mathfrak{T}_4 survives; if $x = 0$ then the first equality above holds since $p_{\mathfrak{T}_4}(0) = 1$ and $P_{\mathfrak{T}_4}(0) = 0$.)

By the fundamental theorem of calculus, $p_{\mathfrak{T}_4}(x)$ is the derivative of $P_{\mathfrak{T}_4}(x)$. Hence, we view (3) as the differential equation that it is. Since $P_{\mathfrak{T}_4}(0) = 0$, by definition the solution of this differential equation is $p_{\mathfrak{T}_4}(x) = \phi(x)$ and $P_{\mathfrak{T}_4}(x) = \Phi(x)$. ■

Let \mathcal{S}_1 (respectively \mathcal{S}_2) be the event that the root of \mathfrak{T}_4 survives under the assumption that its birthtime is equal to $in^{-\varepsilon_1-\varepsilon_2}$ (respectively $(i+1)n^{-\varepsilon_1-\varepsilon_2}$). Let \mathcal{S}_3 (respectively \mathcal{S}_4) be the event that the root of \mathfrak{T}_4 survives, conditioned on the event that its birthtime is at most $in^{-\varepsilon_1-\varepsilon_2}$ (respectively $(i+1)n^{-\varepsilon_1-\varepsilon_2}$), unless $i = 0$ in which case we let \mathcal{S}_3 be the empty event. Let \mathcal{S}_5 be the event that the root of \mathfrak{T}_4 survives, conditioned on the event that its birthtime is in $[in^{-\varepsilon_1-\varepsilon_2}, (i+1)n^{-\varepsilon_1-\varepsilon_2}]$. By Lemma 4.6 and since $\Pr(\mathcal{S}_2) = \Pr(\mathcal{S}_1) \Pr(\mathcal{S}_2 \mid \mathcal{S}_1)$ and $\Pr(\mathcal{S}_4) = \frac{i}{i+1} \Pr(\mathcal{S}_3) + \frac{1}{i+1} \Pr(\mathcal{S}_1) \Pr(\mathcal{S}_5 \mid \mathcal{S}_1)$, we have

$$\Pr(\mathcal{S}_2 \mid \mathcal{S}_1) = \frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \quad \text{and} \quad \Pr(\mathcal{S}_5 \mid \mathcal{S}_1) = \frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})}. \quad (4)$$

Let us consider the events \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_5 . These events depend on the random function β . More accurately, these events depend on the birthtimes of the nodes that are at even distances from the root of \mathfrak{T}_4 . For the purpose of giving a few observations, let us imagine in this paragraph that we can access β through two different oracles. The revealing-oracle reveals everything: given a node ν it returns its birthtime $\beta(\nu)$. The hiding-oracle does not reveal everything: given a node ν it returns its birthtime $\beta(\nu)$ only if its birthtime is at most $in^{-\varepsilon_1-\varepsilon_2}$; otherwise it returns “hidden” (in which case one only learns that $\beta(\nu) > in^{-\varepsilon_1-\varepsilon_2}$). Observe that in order to determine the occurrence of \mathcal{S}_1 , it suffices to only consult the hiding-oracle. In contrast, in order to determine the occurrence of \mathcal{S}_2 and \mathcal{S}_5 , it is not sufficient in general to only consult the hiding-oracle, as these two events may depend on the exact birthtimes of nodes whose birthtimes are larger than $in^{-\varepsilon_1-\varepsilon_2}$; it is, however, sufficient to first consult the hiding-oracle, to verify using the information obtained from the hiding-oracle that \mathcal{S}_1 occurs (this is a necessary condition for the occurrence of both \mathcal{S}_2 and \mathcal{S}_5), and then consult the revealing-oracle for the birthtimes of all nodes whose exact birthtimes were not revealed by the hiding-oracle. The point we’d like to make is that after consulting the hiding-oracle

and verifying that \mathcal{S}_1 occurs, there are some nodes in \mathfrak{T}_4 whose birthtimes need not be queried via the revealing-oracle in order to determine the occurrence of \mathcal{S}_2 and \mathcal{S}_5 . We describe these nodes now. Let ν be a non-root node at even distance from the root of \mathfrak{T}_4 . In order to determine whether or not the grandparent of ν survives, we are interested in knowing (among other things) whether or not the following holds: ν and its siblings all have birthtimes smaller than that of their grandparent, and all survive. From this we get the following observations. If $\beta(\nu) \leq in^{-\varepsilon_1-\varepsilon_2}$ and we know that ν survives given only the information provided by the hiding-oracle, then in order to determine the occurrence of \mathcal{S}_2 and \mathcal{S}_5 , we may ignore the subtree rooted at ν upon querying the revealing-oracle. Further, if $\beta(\nu) \leq in^{-\varepsilon_1-\varepsilon_2}$ and we know that ν does not survive given only the information provided by the hiding-oracle, then in order to determine the occurrence of \mathcal{S}_2 and \mathcal{S}_5 , we may ignore the subtree rooted at the parent of ν (and in particular, ignore the subtrees rooted at ν and its siblings) upon querying the revealing-oracle. Lastly, if $\beta(\nu) > in^{-\varepsilon_1-\varepsilon_2}$ and ν has a child ν_1 , such that given only the information provided by the hiding-oracle we know that for every child ν_2 of ν_1 it holds that $\beta(\nu_2) \leq in^{-\varepsilon_1-\varepsilon_2}$ and ν_2 survives, then in order to determine the occurrence of \mathcal{S}_2 and \mathcal{S}_5 , we may ignore the subtree rooted at the parent of ν upon querying the revealing-oracle. These observations motivate the next definition.

Let \mathfrak{T}_5 be a random rooted tree (depending on the random function β) that is obtained from \mathfrak{T}_4 using the following procedure. For every non-root node ν in \mathfrak{T}_4 at even distance from the root, do: if $\beta(\nu) \leq in^{-\varepsilon_1-\varepsilon_2}$ and ν survives then remove the subtree rooted at ν , and if ν doesn't survive then remove the subtree rooted at the parent of ν ; if $\beta(\nu) > in^{-\varepsilon_1-\varepsilon_2}$, and there is a child ν_1 of ν such that for every child ν_2 of ν_1 it holds that $\beta(\nu_2) \leq in^{-\varepsilon_1-\varepsilon_2}$ and ν_2 survives (which is the same as saying that ν doesn't survive under the assumption that its birthtime is exactly $in^{-\varepsilon_1-\varepsilon_2}$), then remove the subtree rooted at the parent of ν . This gives the random rooted tree \mathfrak{T}_5 . Assign each node ν at even distance from the root of \mathfrak{T}_5 a uniformly random birthtime $\beta'(\nu)$ in the unit interval. Define the event that a node at even distance from the root of \mathfrak{T}_5 survives exactly as it was defined for such a node in \mathfrak{T}_1 , only that in the current definition we replace β with β' . Let \mathcal{S}_6 be the event that the root of \mathfrak{T}_5 survives under the assumption that its birthtime under β' is equal to $n^{-\varepsilon_1-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$. Let \mathcal{S}_7 be the event that the root of \mathfrak{T}_5 survives, conditioned on the event that its birthtime under β' is at most $n^{-\varepsilon_1-\varepsilon_2}/(1 - in^{-\varepsilon_1-\varepsilon_2})$. Given the discussion in the previous paragraph, we observe that $\Pr(\mathcal{S}_2) = \Pr(\mathcal{S}_1 \wedge \mathcal{S}_6)$ and that $\Pr(\mathcal{S}_5) = \Pr(\mathcal{S}_1 \wedge \mathcal{S}_7)$. These two equalities, together with the fact that \mathcal{S}_1 is implied by both \mathcal{S}_2 and \mathcal{S}_5 , give

$$\begin{aligned} \Pr(\mathcal{S}_6 | \mathcal{S}_1) &= \frac{\Pr(\mathcal{S}_2)}{\Pr(\mathcal{S}_1)} = \frac{\Pr(\mathcal{S}_1 \wedge \mathcal{S}_2)}{\Pr(\mathcal{S}_1)} = \Pr(\mathcal{S}_2 | \mathcal{S}_1) \quad \text{and} \\ \Pr(\mathcal{S}_7 | \mathcal{S}_1) &= \frac{\Pr(\mathcal{S}_5)}{\Pr(\mathcal{S}_1)} = \frac{\Pr(\mathcal{S}_1 \wedge \mathcal{S}_5)}{\Pr(\mathcal{S}_1)} = \Pr(\mathcal{S}_5 | \mathcal{S}_1). \end{aligned} \tag{5}$$

From (4) and (5) we get

$$\Pr(\mathcal{S}_6 | \mathcal{S}_1) = \frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \quad \text{and} \quad \Pr(\mathcal{S}_7 | \mathcal{S}_1) = \frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})}. \tag{6}$$

In addition, by (5), Lemma 4.6 (which implies $\Pr(\mathcal{S}_2) = \phi((i+1)n^{-\varepsilon_1})$), Lemma 3.1 (which implies $\phi((i+1)n^{-\varepsilon_1}) \geq n^{-\Theta(\varepsilon_1)}$), and the fact that \mathcal{S}_2 implies \mathcal{S}_5 ,

$$\begin{aligned} \Pr(\mathcal{S}_6 | \mathcal{S}_1) &\geq \Pr(\mathcal{S}_2) \geq n^{-\Theta(\varepsilon_1)} \quad \text{and} \\ \Pr(\mathcal{S}_7 | \mathcal{S}_1) &\geq \Pr(\mathcal{S}_5) \geq \Pr(\mathcal{S}_2) \geq n^{-\Theta(\varepsilon_1)}. \end{aligned} \tag{7}$$

Let \mathcal{S}_8 be the event that for every node at even distance less than $2c$ from the root of \mathfrak{T}_5 , the number of children of that node which in turn have exactly j children is in $z_{i,j}(1 \pm 3000\Gamma_i)$. By Lemma 4.6, Chernoff's bound and the union bound, one can find that $\Pr(\mathcal{S}_8) \geq 1 - n^{-\omega(1)}$. This, with (7), and the fact that $\Pr(\mathcal{S}_1) \geq n^{-\Theta(\varepsilon_1)}$ (which follows from Lemmas 4.6 and 3.1), implies

$$\begin{aligned} \Pr(\mathcal{S}_6 | \mathcal{S}_1 \wedge \mathcal{S}_8) &\in \Pr(\mathcal{S}_6 | \mathcal{S}_1)(1 \pm n^{-\omega(1)}) \quad \text{and} \\ \Pr(\mathcal{S}_7 | \mathcal{S}_1 \wedge \mathcal{S}_8) &\in \Pr(\mathcal{S}_7 | \mathcal{S}_1)(1 \pm n^{-\omega(1)}). \end{aligned} \tag{8}$$

To conclude the proof, note that if we condition on $\mathcal{S}_1 \wedge \mathcal{S}_8$, \mathfrak{T}_5 is a random tree which is isomorphic to a tree that satisfies the same properties as \mathfrak{T}_2 . Hence, conditioned on $\mathcal{S}_1 \wedge \mathcal{S}_8$, the probability of \mathcal{S}_6 is a weighted average of elements in $\bigcup_{\mathfrak{T}'_2} \{p_{\mathfrak{T}'_2}(n^{-\varepsilon_1})\}$, and the probability of \mathcal{S}_7 is a weighted average of elements in $\bigcup_{\mathfrak{T}'_2} \{n^{\varepsilon_1} P_{\mathfrak{T}'_2}(n^{-\varepsilon_1})\}$, where $\bigcup_{\mathfrak{T}'_2}$ ranges over trees that satisfy the same properties as \mathfrak{T}_2 . By applying Lemma 4.2 twice we get that $p_{\mathfrak{T}'_2}(n^{-\varepsilon_1}) \in p_{\mathfrak{T}_2}(n^{-\varepsilon_1})(1 \pm 11\Gamma_i\gamma_i)$ for every tree \mathfrak{T}'_2 that satisfies the same properties as \mathfrak{T}_2 . Therefore, $\Pr(\mathcal{S}_6 | \mathcal{S}_1 \wedge \mathcal{S}_8) \in p_{\mathfrak{T}_2}(n^{-\varepsilon_1})(1 \pm 11\Gamma_i\gamma_i)$. Hence, $p_{\mathfrak{T}_2}(n^{-\varepsilon_1}) \in \Pr(\mathcal{S}_6 | \mathcal{S}_1 \wedge \mathcal{S}_8)(1 \pm 12\Gamma_i\gamma_i)$. A similar argument shows that $P_{\mathfrak{T}_2}(n^{-\varepsilon_1}) \in n^{-\varepsilon_1} \Pr(\mathcal{S}_7 | \mathcal{S}_1 \wedge \mathcal{S}_8)(1 \pm 12\Gamma_i\gamma_i)$. This, together with (6) and (8), implies

$$\begin{aligned} p_{\mathfrak{T}_2}(n^{-\varepsilon_1}) &\in \frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})}(1 \pm 13\Gamma_i\gamma_i) \quad \text{and} \\ P_{\mathfrak{T}_2}(n^{-\varepsilon_1}) &\in \frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})}(1 \pm 13\Gamma_i\gamma_i). \end{aligned}$$

These estimates, together Lemma 4.2, give Lemma 4.1.

5 Survival and the process

In this section we relate the main result of the previous section to the process. Fix for the rest of the section an integer $0 \leq i < I$. For an integer $1 \leq j \leq 5$ and an edge $f \in \text{NotTraversed}_i$, let $X''_{i,j}(f)$ be the set of all $G \in X_{i,j}(f)$ such that $G \subseteq M_i \cup \text{BigBite}_{i+1}$. For an integer $c \geq 1$ and an edge $f \in \text{NotTraversed}_i$, we define a finite, rooted, labeled tree $\mathfrak{T}_c(f)$; to do so, we first define another tree $\mathfrak{T}'_c(f)$ and then alter it to obtain $\mathfrak{T}_c(f)$. Let $\mathfrak{T}'_c(f)$ be the finite, rooted, labeled tree with the following four properties: first, every leaf in the tree is at distance $2c$ from the root; second, the root is labeled with the edge f ; third, if a non-leaf node ν at even distance from the root is labeled with an edge g , then its set of children is the set $\{\nu_G : G \in \bigcup_{1 \leq j \leq 5} X''_{i,j}(g)\}$, where the label of ν_G is the graph G ; fourth, if a node ν at odd distance from the root is labeled with a graph G , then its set of children is the set $\{\nu_g : g \in G \cap \text{BigBite}_{i+1}\}$, where the label of ν_g is the edge g . Let $\mathfrak{T}_c(f)$ be obtained by removing subtrees from $\mathfrak{T}'_c(f)$ as follows: for every non-leaf, non-root node ν at even distance from the root, if ν has a child labeled G and a grandparent labeled $g \in G$, then remove the subtree rooted at the child labeled G .

Let ν_0 be a node labeled g_0 at even distance from the root of $\mathfrak{T}_c(f)$. Define the event that ν_0 survives as follows. If ν_0 is a leaf then ν_0 survives by definition. Otherwise, ν_0 survives if and only if for every child ν_1 of ν_0 , the following holds: if for every child ν_2 of ν_1 , labeled g_2 , we have $g_2 \in \text{Bite}_{i+1}$, and in case $g_0 \in \text{Bite}_{i+1}$ we also have that the birthtime of g_2 is less than the birthtime of g_0 , then ν_1 has a child that does not survive. For $f \in \text{NotTraversed}_i$, let $\mathcal{S}_c(f)$ be the event

that the root of $\mathfrak{T}_c(f)$ survives. For $F \subseteq \text{NotTraversed}_i$, let $\mathcal{S}_c(F) := \bigwedge_{f \in F} \mathcal{S}_c(f)$, and let $\mathcal{I}_c(F)$ be the event that every two distinct nodes at even distances from the roots of the trees in the forest $\{\mathfrak{T}_c(f) : f \in F\}$ have two distinct labels.

Lemma 5.1. *Let $c \geq 1$ be an odd integer. Let $F \subseteq \text{NotTraversed}_i$ and assume that $M_i \cup F$ is K_4 -free. Then: (i) assuming $|F| = 1$, $\mathcal{S}_c(F) \implies M_{i+1} \cup F$ is K_4 -free $\implies \mathcal{S}_{c+1}(F)$; (ii) assuming $|F| \geq 2$, $\mathcal{I}_c(F) \wedge \mathcal{S}_c(F) \implies M_{i+1} \cup F$ is K_4 -free.*

Proof. Let $c \geq 1$ be an odd integer, let $F \subseteq \text{NotTraversed}_i$ and assume that $M_i \cup F$ is K_4 -free. The second item follows directly from the first item. So it remains to prove the first item. For that purpose, assume for the rest of the proof that $F = \{f\}$. We need the following claim.

Claim 5.2. *Let $b \geq 2$ be an even integer. Let ν_0 be a node labeled g_0 at height $2b$ in $\mathfrak{T}_c(f)$ or in $\mathfrak{T}_{c+1}(f)$. If ν_0 doesn't survive then $M_{i+1} \cup \{g_0\}$ is not K_4 -free.*

Proof. The proof is by induction on b . We start with a general setup that applies both to the base case and to the induction step. Let $b \geq 2$ be an even integer. Let ν_0 be a node labeled g_0 at height $2b$ in $\mathfrak{T}_c(f)$ or in $\mathfrak{T}_{c+1}(f)$. Assume that ν_0 doesn't survive. We show that there is a node ν_1 labeled G_1 , which is a child of ν_0 , such that $G_1 \subseteq M_{i+1}$. This will give us that $M_{i+1} \cup \{g_0\}$ is not K_4 -free. Since ν_0 doesn't survive we have that there is a node ν_1 labeled G_1 , which is a child of ν_0 , such that for every child ν_2 of ν_1 , letting g_2 be the label of ν_2 , the following two properties hold: first, $g_2 \in \text{Bite}_{i+1}$ and if $g_0 \in \text{Bite}_{i+1}$ then the birthtime of g_2 is less than the birthtime of g_0 ; second, ν_2 survives. Fix such a child ν_1 of ν_0 . It remains to show that for every child ν_2 of ν_1 , letting g_2 be the label of ν_2 , the fact that ν_2 survives implies $M_{i+1} \cup \{g_2\}$ is K_4 -free, as we already know that $g_2 \in \text{Bite}_{i+1}$. This will give us that $G_1 \subseteq M_{i+1}$. So let us fix such a child ν_2 of ν_1 . To show that $M_{i+1} \cup \{g_2\}$ is K_4 -free we need to show that for every $G_3 \in \bigcup_{1 \leq j \leq 5} X''_{i,j}(g_2)$, either there is an edge in G_3 whose birthtime is larger than that of g_2 , or otherwise $G_3 \not\subseteq M_{i+1}$. Fix a graph $G_3 \in \bigcup_{1 \leq j \leq 5} X''_{i,j}(g_2)$. Then either G_3 is a label of a child of ν_2 or $g_0 \in G_3$, and this follows from the definition of $\mathfrak{T}_c(f)$ and $\mathfrak{T}_{c+1}(f)$. If $g_0 \in G_3$ then the birthtime of g_0 is larger than the birthtime of g_2 and we are done. So assume that G_3 is a label of a child ν_3 of ν_2 . This is where the arguments for the base case and the induction step differ.

For the base case, assume that $b = 2$. Then every child of ν_3 is a leaf. Since a leaf survives by definition, the fact that ν_2 survives implies that ν_3 either has a child whose label is not in Bite_{i+1} , in which case $G_3 \not\subseteq M_{i+1}$ as needed, or ν_3 has a child whose label has a birthtime larger than that of g_2 , as needed. For the induction step, assume that $b \geq 4$ and that the claim holds for $b - 2$. If ν_2 survives then either ν_3 has a child whose label is not in Bite_{i+1} , in which case $G_3 \not\subseteq M_{i+1}$ as needed; or ν_3 has a child whose label has a birthtime larger than that of g_2 , as needed; or ν_3 has a child that doesn't survive, in which case, by the induction hypothesis, for some $g_4 \in G_3$, $M_{i+1} \cup \{g_4\}$ is not K_4 -free, and so $G_3 \not\subseteq M_{i+1}$ as needed. ■

Since $c + 1$ is even, it follows from the claim above that $M_{i+1} \cup F$ is K_4 -free $\implies \mathcal{S}_{c+1}(F)$. Since c is odd, it also follows from the claim above and from the definition of $\mathcal{S}_c(F)$, that if $\mathcal{S}_c(F)$ holds then the following holds: for every node ν whose label is G and which is a child of the root of $\mathfrak{T}_c(f)$, $G \not\subseteq M_{i+1}$. Since $M_i \cup F$ is K_4 -free and since every graph $G \in \bigcup_{1 \leq j \leq 5} X''_{i,j}(f)$ is a label of a child of the root of $\mathfrak{T}_c(f)$, we get that $\mathcal{S}_c(F) \implies M_{i+1} \cup F$ is K_4 -free. ■

For an edge $f \in \text{NotTraversed}_i$ and a set $R \subseteq [n]$, let $\mathfrak{T}_c(f, R)$ be obtained by removing subtrees from $\mathfrak{T}_c(f)$ as follows: for every child ν of the root of $\mathfrak{T}_c(f)$, if ν is labeled with a graph that shares at least three vertices with R , then remove the subtree rooted at ν . Define the event that a node at even distance from the root of $\mathfrak{T}_c(f, R)$ survives exactly as it was defined for such a node in $\mathfrak{T}_c(f)$. Let $\mathcal{S}_c(f, R)$ be the event that the root of $\mathfrak{T}_c(f, R)$ survives. For two disjoint graphs $F_1, F_2 \subseteq \text{NotTraversed}_i$, let $\mathcal{S}_c(F_1, F_2, R)$ be the event $[\bigwedge_{f \in F_1} \mathcal{S}_c(f)] \wedge [\bigwedge_{f \in F_2} \mathcal{S}_c(f, R)]$, and let $\mathcal{I}_c(F_1, F_2, R)$ be the event that every two distinct nodes at even distances from the roots of the trees in the forest $\{\mathfrak{T}_c(f) : f \in F_1\} \cup \{\mathfrak{T}_c(f, R) : f \in F_2\}$ have two distinct labels.

Lemma 5.3. *Let $c \geq 1$ be an odd integer. Let $F_1, F_2 \subseteq \text{NotTraversed}_i$ be two disjoint graphs and assume that $M_i \cup F_1 \cup F_2$ is K_4 -free. Let $R \subseteq [n]$. Then: (i) $\mathcal{I}_c(F_1 \cup F_2) \wedge \mathcal{S}_c(F_1 \cup F_2) \implies \mathcal{I}_c(F_1, F_2, R) \wedge \mathcal{S}_c(F_1, F_2, R)$; (ii) $\mathcal{I}_c(F_1, F_2, R) \wedge \mathcal{S}_c(F_1, F_2, R)$ implies that $M_{i+1} \cup F_1$ is K_4 -free, and for every edge $f \in F_2$, if $|X_{i+1,0}(f)| > 0$ (meaning $M_{i+1} \cup \{f\}$ is not K_4 -free), then for every $G \in X_{i+1,0}(f)$, G shares at least three vertices with R .*

Proof. The first item follows from the definition of the underlying events. The second item can be proved using the same argument used in the proof of Lemma 5.1, and in particular, using Claim 5.2. \blacksquare

Fix for the rest of the section a graph $F \subseteq \text{NotTraversed}_i$, with $1 \leq |F| = a_1 + a_2 \leq 3$, and such that $M_i \cup F$ is K_4 -free. Also, fix an integer $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$, and note that c is a sufficiently large constant. Let \mathcal{E}_0 be the event that $|F \cap \text{Bite}_{i+1}| = a_1$. Let \mathcal{E}_1 be the event that for every $f \in F$, the set of children of every non-leaf node at even distance from the root of $\mathfrak{T}_c(f)$ can be partitioned to 5 sets of children, where the j th set satisfies the following: it consists of nodes whose labels have exactly j edges in BigBite_{i+1} , and it has size in $z_{i,j}(1 \pm 3000\Gamma_i)$. For brevity, set $\mathcal{E}_2 = \mathcal{I}_c(F)$.

The next lemma is the main result of this section.

Lemma 5.4. *Assume that M_i and BIGBite_{i+1} are given so that $\mathcal{C}_i \wedge \mathcal{D}_i$ holds. Then*

$$\Pr(\mathcal{S}_c(F) | \mathcal{E}_0) \in \left(\frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})} \right)^{a_1} \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^{a_2} (1 \pm 90\Gamma_i\gamma_i)$$

and

$$\Pr(\mathcal{E}_2 \wedge \mathcal{S}_c(F) | \mathcal{E}_0) \in \Pr(\mathcal{S}_c(F) | \mathcal{E}_0)(1 \pm o(\Gamma_i\gamma_i)),$$

where the probabilities are both over the choice of BigBite_{i+1} , Bite_{i+1} and the choice of the birthtimes of the edges in Bite_{i+1} .

Let us prove Lemma 5.4. To this end, assume for the rest of the section that M_i and BIGBite_{i+1} are given so that $\mathcal{C}_i \wedge \mathcal{D}_i$ holds. In the two subsections below we will prove that

$$\Pr(\mathcal{E}_1 | \mathcal{E}_0) \geq 1 - n^{-\omega(1)} \quad \text{and} \quad (9)$$

$$\Pr(\mathcal{E}_2 | \mathcal{E}_0) \geq 1 - n^{-\Theta(\varepsilon_3)}. \quad (10)$$

Here we use these estimates to prove the lemma.

We start by obtaining an estimate for $\Pr(\mathcal{S}_c(F) | \mathcal{E}_0)$. First, observe that (9) and (10), together with Lemma 3.1 and the fact that ε_1 is sufficiently small with respect to ε_3 , imply that

$$\Pr(\mathcal{E}_1 \wedge \mathcal{E}_2 | \mathcal{E}_0) \geq 1 - o(\Gamma_i \gamma_i).$$

Second, note that if we condition on $\mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2$, then every tree in $\{\mathfrak{T}_c(f) : f \in F\}$ satisfies the same properties that are satisfied by the tree \mathfrak{T}_1 , the tree that was studied in the previous section, and the events in $\{\mathcal{S}_c(f) : f \in F\}$ are mutually independent. Hence, under this condition we can use the main result of the previous section, Lemma 4.1, to find that

$$\Pr(\mathcal{S}_c(F) | \mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2) \in \left(\frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1} \phi(in^{-\varepsilon_1})} \right)^{a_1} \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^{a_2} (1 \pm 80\Gamma_i \gamma_i).$$

Third, observe that $\Pr(\mathcal{S}_c(F) | \mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2) = \Omega(1)$. (Indeed, a sufficient condition for $\mathcal{S}_c(F)$ is that for every $f \in F$, for every child ν_1 of the root of $\mathfrak{T}_c(f)$ there is a child ν_2 whose label is not in $Bite_{i+1}$. Assuming $\mathcal{E}_1 \wedge \mathcal{E}_2$ this event occurs with probability $\Omega(1)$.) Since $\Pr(\mathcal{S}_c(F) | \mathcal{E}_0) = \Pr(\mathcal{E}_1 \wedge \mathcal{E}_2 | \mathcal{E}_0) \Pr(\mathcal{S}_c(F) | \mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2) + O(\Pr(\neg(\mathcal{E}_1 \wedge \mathcal{E}_2) | \mathcal{E}_0))$, the above three facts give the desired estimate for $\Pr(\mathcal{S}_c(F) | \mathcal{E}_0)$.

To obtain an estimate for $\Pr(\mathcal{E}_2 \wedge \mathcal{S}_c(F) | \mathcal{E}_0)$ and complete the proof, simply note that

$$\begin{aligned} \Pr(\mathcal{S}_c(F) | \mathcal{E}_0) &\geq \Pr(\mathcal{E}_2 \wedge \mathcal{S}_c(F) | \mathcal{E}_0) \\ &\geq \Pr(\mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{S}_c(F) | \mathcal{E}_0) \\ &\geq \Pr(\mathcal{E}_1 \wedge \mathcal{E}_2 | \mathcal{E}_0) \cdot \Pr(\mathcal{S}_c(F) | \mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2), \end{aligned}$$

and apply our findings from above.

5.1 Proof of (9)

Since clearly $\Pr(\mathcal{E}_0) \geq n^{-\Theta(1)}$, it suffices to prove that $\Pr(\mathcal{E}_1) \geq 1 - n^{-\omega(1)}$. Let $f \in F$, let ν be a non-leaf node labeled g at even distance from the root of $\mathfrak{T}_c(f)$, and let $1 \leq j \leq 5$. Note that by the union bound it is enough to prove that each of the following two properties occurs with probability at least $1 - n^{-\omega(1)}$: first, the number of children of ν which are labeled with a graph $G \in X''_{i,j}(g)$ is equal to $|X''_{i,j}(g)|$, up to an additive factor of $(\ln n)^{O(1)}$; second, $|X''_{i,j}(g)| \in z_{i,j}(1 \pm 2999\Gamma_i)$.

To show that the first property occurs with probability at least $1 - n^{-\omega(1)}$, recall the definition of $\mathfrak{T}_c(f)$ and observe that it suffices to show that with probability at least $1 - n^{-\omega(1)}$, for every three vertices $v_1, v_2, v_3 \in [n]$, there are at most $(\ln n)^{O(1)}$ other vertices in $[n]$ that are adjacent in $Traversed_i \cup BigBite_{i+1}$ simultaneously to v_1, v_2 and v_3 . Indeed, by the fact that $BigBite_{i+1} \subseteq BIGBite_{i+1}$ and by (C3), the above occurs with probability 1.

Next, we show that the second property occurs with probability at least $1 - n^{-\omega(1)}$. For that we assume that either $i \geq 1$, or else $j = 5$, since otherwise trivially $|X''_{i,j}(f)| = z_{i,j} = 0$ and we are done. Let $\{G'_l : l \in L\}$ be the set $X'_{i,j}(g)$. By (D2) we have $|L| = |X'_{i,j}(g)| \in n^{(\varepsilon_3 - 2/5)j} x_{i,j} (1 \pm 2000\Gamma_i)$. Let $\{G_l : l \in L\}$ be the family (potentially a multiset) for which it holds that $G_l = G'_l \cap BIGBite_{i+1}$ for every $l \in L$. Consider the binomial random graph $G(n, p)$ with $p = n^{\varepsilon_2 - \varepsilon_3}$, and let W be as defined at the beginning of Section 2. Note that $\mathbb{E}(W) = n^{(\varepsilon_2 - \varepsilon_3)j} |L| \in z_{i,j} (1 \pm 2000\Gamma_i)$. Also note that W has the same distribution as $|X''_{i,j}(g)|$. It remains to argue that the probability that W

deviates from its expectation by more than $999\Gamma_i z_{i,j} \geq \mathbb{E}(W)^{0.9} = n^{\Omega(\varepsilon_2)}$ is at most $n^{-\omega(1)}$. To do so, note that if $1 \leq |G| \leq j$ then $|L_G| \leq (\ln n)^{O(1)}$, and that this follows from (C3). Apply Theorem 2.1 with $\mathfrak{E}_0 = \mathbb{E}(W)$, $\mathfrak{E}_k = \exp((2j-k)\sqrt{\ln n})$ for $1 \leq k \leq j$, and $\lambda = (\ln n)^2$.

5.2 Proof of (10)

Say that a sequence $(G_l)_{l=1}^m$ is bad, if the following properties hold:

- $1 \leq m \leq 2c$;
- for all $1 \leq l \leq m$, $G_l \in \bigcup_{1 \leq j \leq 5} X'_{i,j}(g)$ for some $g \in F \cup \bigcup_{k < l} (G_k \cap \text{BIGBite}_{i+1})$;
- for all $1 \leq l < m$, $G_l \cap \text{BIGBite}_{i+1}$ shares no edge with $F \cup \bigcup_{k < l} (G_k \cap \text{BIGBite}_{i+1})$;
- one of the following holds:
 - $G_m \cap \text{BIGBite}_{i+1}$ shares at least 1 edge but not all edges with $F \cup \bigcup_{k < m} (G_k \cap \text{BIGBite}_{i+1})$;
 - $G_m \cap \text{BIGBite}_{i+1}$ shares no edge with $F \cup \bigcup_{k < m} (G_k \cap \text{BIGBite}_{i+1})$, and there is a vertex outside of the vertex set of G_m that is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ to at least three vertices of G_m ;
 - $G_m \cap \text{BIGBite}_{i+1}$ shares no edge with $F \cup \bigcup_{k < m} (G_k \cap \text{BIGBite}_{i+1})$. Moreover, let g_m be such that $G_m \in \bigcup_{1 \leq j \leq 5} X'_{i,j}(g_m)$. Then there is a vertex of G_m that is not a vertex of g_m , and which is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1} \cup F$ to at least three vertices of $F \cup \bigcup_{k < m} G_k$.

Note that for every bad sequence it holds that its members are all contained in $M_i \cup \text{BIGBite}_{i+1}$. Let \mathcal{E}_3 be the event that there is no bad sequence whose members are all contained in $M_i \cup \text{BIGBite}_{i+1}$. The next two lemmas imply (10).

Lemma 5.5. $\mathcal{E}_3 \implies \mathcal{E}_2$.

Proof. We prove the contrapositive. Assume $\neg \mathcal{E}_2$ holds and consider the forest $\{\mathcal{T}_c(f) : f \in F\}$. Then for some $1 \leq m \leq 2c$ there is a sequence $(\nu_l)_{l=1}^m$ of nodes at odd distances from the roots in the forest such that, denoting by G_l the label of ν_l , the following holds: for all $1 \leq l \leq m$, ν_l is either a child of a root in the forest, or a grandchild of some ν_k with $k < l$; furthermore, for all $1 \leq l < m$, $G_l \cap \text{BigBite}_{i+1}$ shares no edge with $F \cup \bigcup_{k < l} (G_k \cap \text{BigBite}_{i+1})$, while $G_m \cap \text{BigBite}_{i+1}$ shares at least 1 edge with $F \cup \bigcup_{k < m} (G_k \cap \text{BigBite}_{i+1})$. Let g_l be the label of the parent of ν_l . Note that for all $1 \leq l \leq m$, $g_l \in F \cup \bigcup_{k < l} (G_k \cap \text{BigBite}_{i+1})$, $G_l \in \bigcup_{1 \leq j \leq 5} X'_{i,j}(g_l)$, and $G_l \cap \text{BigBite}_{i+1} = G_l \cap \text{BIGBite}_{i+1}$. Hence, every non-empty prefix of $(G_l)_{l=1}^m$ satisfies the first three properties of a bad sequence. By definition, the members of $(G_l)_{l=1}^m$ are all contained in $M_i \cup \text{BigBite}_{i+1}$, and so in order to conclude that $\neg \mathcal{E}_3$ holds it remains to show that some non-empty prefix of $(G_l)_{l=1}^m$ satisfies the fourth property of a bad sequence.

Since $G_m \cap \text{BigBite}_{i+1}$ shares at least 1 edge with $F \cup \bigcup_{k < m} (G_k \cap \text{BigBite}_{i+1})$, we may assume that $G_m \cap \text{BigBite}_{i+1}$ shares all of its edges with $F \cup \bigcup_{k < m} (G_k \cap \text{BigBite}_{i+1})$, since otherwise $(G_l)_{l=1}^m$ satisfies the fourth property of a bad sequence and we are done.

Suppose that $G_m \cap \text{BigBite}_{i+1} \subseteq F$. By assumption, $M_i \cup F$ is K_4 -free and so since $G_m \subseteq M_i \cup \text{BigBite}_{i+1}$, we must have that $g_m \notin F$. Hence, there exists $1 \leq m' < m$ such that $g_m \in G_{m'}$. We claim that $(G_l)_{l=1}^{m'}$ satisfies the fourth property of a bad sequence. Indeed, let $f \in G_m \cap \text{BigBite}_{i+1} \subseteq F$. By the definition of the trees in the forest, $g_{m'} \notin G_m$ and so $f \neq g_{m'}$. Also, $G_{m'} \cap \text{BigBite}_{i+1}$ shares no edge with F and so $f \notin G_{m'}$. It follows that there is a vertex of $G_{m'}$ (more accurately, a vertex of g_m) that is not a vertex of $g_{m'}$, and which is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ to at least three vertices of $F \cup \bigcup_{k < m'} G_k$ (these three vertices being the two vertices of $g_{m'}$ and one vertex of f).

Suppose that $G_m \cap \text{BigBite}_{i+1} \not\subseteq F$. Then there exists $1 \leq m' < m$ such that $G_m \cap \text{BigBite}_{i+1}$ and $G_{m'} \cap \text{BigBite}_{i+1}$ share some edge g , and such that m' is maximal with respect to that property. If ν_m and $\nu_{m'}$ are siblings, then it is clear that there is a vertex outside of the vertex set of $G_{m'}$ that is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ to at least three vertices of $G_{m'}$. If ν_m is a grandchild of $\nu_{m'}$, then since by the definition of the trees in the forest $g_{m'} \notin G_m$, we have that there is a vertex outside of the vertex set of $G_{m'}$ that is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ to at least three vertices of $G_{m'}$. Therefore, if ν_m and $\nu_{m'}$ are siblings or if ν_m is a grandchild of $\nu_{m'}$, then $(G_l)_{l=1}^{m'}$ satisfies the fourth property of a bad sequence, and we are done. So assume that ν_m and $\nu_{m'}$ are not siblings, and that ν_m is not a grandchild of $\nu_{m'}$. Then since $(G_l)_{l=1}^m$ satisfies the first three properties of a bad sequence, we have that $g_m \neq g_{m'}$ and $g_m \notin G_{m'}$. Now, we have two cases: either $g_m \in F \cup \bigcup_{k < m'} (G_k \cap \text{BigBite}_{i+1})$, or $g_m \in \bigcup_{m' < k < m} (G_k \cap \text{BigBite}_{i+1})$. If the first case holds, since $g_m \neq g_{m'}$, we have that there is a vertex of $G_{m'}$ (more accurately, a vertex of g) that is not a vertex of $g_{m'}$, and which is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1} \cup F$ to at least three vertices of $F \cup \bigcup_{k < m'} G_k$ (these three vertices being the two vertices of $g_{m'}$ and one vertex of g_m). If the second case holds, since $g_m \notin G_{m'}$, we have that $g_m \in \bigcup_{m' < k < m} (G_k \cap \text{BigBite}_{i+1})$. In that case, $g_m \in G_{m''} \cap \text{BigBite}_{i+1}$, for some $m' < m'' < m$. Since $g \in G_m \cap \text{BigBite}_{i+1}$ and $G_m \in \bigcup_{1 \leq j \leq 5} X'_{i,j}(g_m)$, by the definition of the trees in the forest, $g \neq g_{m''}$. Also, by the maximality of m' , $g \notin G_{m''}$. Hence, g has a vertex outside of the vertex set of $G_{m''}$. Note that $g \in F \cup \bigcup_{k < m''} (G_k \cap \text{BigBite}_{i+1})$. It follows that there is a vertex of $G_{m''}$ (more accurately, a vertex of g_m) that is not a vertex of $g_{m''}$, and which is adjacent in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ to at least three vertices of $F \cup \bigcup_{k < m''} G_k$ (these three vertices being the two vertices of $g_{m''}$ and one vertex of g). \blacksquare

Lemma 5.6. $\Pr(\mathcal{E}_3 \mid \mathcal{E}_0) \geq 1 - n^{-\Theta(\varepsilon_3)}$.

Proof. Fix $1 \leq m \leq 2c$ and note that $m = O(1)$. By the union bound, it is enough to show that conditioned on \mathcal{E}_0 , the expected number of bad sequences of length m whose members are all contained in $M_i \cup \text{BigBite}_{i+1}$ is at most $n^{-\Theta(\varepsilon_3)}$.

Say that a sequence $(G_l)_{l=1}^{m-1}$ is almost-bad if there exists G_m such that $(G_l)_{l=1}^m$ is bad. We first claim that conditioned on \mathcal{E}_0 , the expected number of almost-bad sequences whose members are all contained in $M_i \cup \text{BigBite}_{i+1}$ is at most $n^{O(\varepsilon_2 m)}$, which is at most $n^{O(\varepsilon_3^2)}$, since $m \leq 2c \leq 4\varepsilon_3^2 \varepsilon_2^{-1}$. This claim follows from the definition of a bad sequence, the fact that (D2) holds (specifically the fact that $|X'_{i,j}(f)| \leq n^{\varepsilon_3 j + o(1)}$ for all $1 \leq j \leq 5$ and all $f \in \text{NotTraversed}_i$), and since the probability that $G \in X'_{i,j}(f)$ is contained in $M_i \cup \text{BigBite}_{i+1}$ is $n^{(\varepsilon_2 - \varepsilon_3)j}$.

Given an almost-bad sequence $(G_l)_{l=1}^{m-1}$, the number of graphs $G_m \subseteq M_i \cup \text{BIGBite}_{i+1}$ that can be concatenated to this sequence so that it becomes bad is at most $(\ln n)^{O(1)}$, and this follows

from the definition of a bad sequence together with (C3) and (C4). For every almost-bad sequence $(G_l)_{l=1}^{m-1}$ and every graph $G_m \subseteq M_i \cup \text{BIGBite}_{i+1}$ that can be concatenated to this sequence so that it becomes bad, there is an edge in $G_m \cap \text{BIGBite}_{i+1}$ which does not belong to F , nor to any of the graphs in the almost-bad sequence. Hence, conditioned on the event that every member of a given almost-bad sequence is contained in $M_i \cup \text{BIGBite}_{i+1}$, and conditioned on \mathcal{E}_0 , the probability that a given G_m as above is contained in $M_i \cup \text{BIGBite}_{i+1}$ is at most $n^{\varepsilon_2 - \varepsilon_3}$. It follows that the expected number of bad sequences of length m is at most $n^{O(\varepsilon_3^2)} \cdot (\ln n)^{O(1)} \cdot n^{\varepsilon_2 - \varepsilon_3}$, which is at most $n^{-\Theta(\varepsilon_3)}$. ■

6 Supporting lemmas

In this section we prove two supporting lemmas that will be used in the next section, where we prove our main lemma. Fix for the rest of the section an integer $0 \leq i < I$. We start with the following lower bound on the probability of \mathcal{C}_i .

Lemma 6.1. $\Pr(\mathcal{C}_i) \geq 1 - n^{-\omega(1)}$.

Proof. We argue that every property that is asserted to hold by \mathcal{C}_i occurs with probability at least $1 - n^{-\omega(1)}$. The key observation here is that $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ is the binomial random graph $G(n, p)$, for some $p = \Theta(n^{\varepsilon_3 - 2/5})$ that we fix for the rest of the proof. With that observation at hand, we continue as follows. A standard application of Chernoff's bound and the union bound shows that (C1), (C2) and (C3), each occurs with probability at least $1 - n^{-\omega(1)}$. Theorem 2.1 and the union bound easily implies that (C4) and (C5), each occurs with probability at least $1 - n^{-\omega(1)}$. Theorem 2.3 and the union bound easily implies that (C6) occurs with probability at least $1 - n^{-\omega(1)}$. To complete the proof, we need to show that (C7) and (C8), each occurs with probability at least $1 - n^{-\omega(1)}$. For that, we need the following claim.

Claim 6.2. *Let $R \subseteq [n]$ be a set of r vertices, where $s - o(s) \leq r \leq s$. Let Q be a set containing at most q paths of length two in $\binom{R}{2}$. With probability at least $1 - n^{-\omega(s)}$, the following holds for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$: the number of paths in Q that are contained in $M \cap \binom{R}{2}$ is at most $O(qn^{-3.99/5} + n^{4.2/5})$.*

Proof. The expected number of paths in Q that are contained in $G(n, p)$ is at most $qp^2 \leq qn^{-3.99/5}$. Hence, by Theorem 2.3, with probability at least $1 - n^{-\omega(s)}$, there is a set $E_0 \subseteq G(n, p)$ of size at most $n^{3.1/5}$, such that $G(n, p) \setminus E_0$ contains fewer than $2qn^{-3.99/5}$ paths from Q . Moreover, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, every edge in E_0 belongs to at most $2n^{1.1/5}$ paths of length two in $M \cap \binom{R}{2}$. Therefore, with the desired probability, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, the number of paths in Q that are contained in $M \cap \binom{R}{2}$ is at most $2qn^{-3.99/5} + |E_0| \cdot 2n^{1.1/5} = O(qn^{-3.99/5} + n^{4.2/5})$. ■

Property (C7). Fix a set $R \subseteq [n]$ of r vertices, where $s - o(s) \leq r \leq s$. To show that (C7) occurs with probability at least $1 - n^{-\omega(1)}$, it is enough to show that R satisfies the first assertion of (C7) with probability at least $1 - n^{-\omega(s)}$ and the second assertion of (C7) with probability at least $1 - n^{-\omega(s)}$.

We begin with the first assertion. We show that with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, there are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq M$, which shares all four vertices with R . For that it is enough to show that with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, there are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ whose two vertices form an independent set in some 4-cycle in $M \cap \binom{R}{2}$. Note that the expected number of 4-cycles in $G(n, p) \cap \binom{R}{2}$ is at most $n^{4.2/5}$. Hence, by Theorem 2.3, with probability at least $1 - n^{-\omega(s)}$, there is a set $E_0 \subseteq G(n, p)$ of size at most $n^{3.1/5}$, such that $(G(n, p) \cap \binom{R}{2}) \setminus E_0$ contains fewer than $2n^{4.2/5}$ 4-cycles. Also, every 4-cycle has two independent sets of size two. Hence, with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, the number of edges $g \in \binom{R}{2}$ whose two vertices form an independent set in some 4-cycle in $M \cap \binom{R}{2}$ is at most $4n^{4.2/5} + |E_0| \cdot 2n^{1.1/5} = O(n^{4.2/5})$.

We continue with the second assertion. We show that with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, there is a set $R_0 \subseteq [n] \setminus R$ of at most $n^{0.99/5}$ vertices, such that there are at most $O(n^{4.2/5})$ edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq M$, which shares exactly three vertices with R and one vertex with $[n] \setminus (R \cup R_0)$. Start by exposing only the edges in $G(n, p) \setminus \binom{R}{2}$. It is easy to show that with probability at least $1 - n^{-\omega(s)}$, we can partition the set of vertices $[n] \setminus R$ to two sets, R_0 and R_1 , where R_0 has size at most $n^{0.99/5}$, where every vertex in R_1 is adjacent in $G(n, p)$ to at most $n^{2.02/5}$ vertices in R , and where the number of vertices in R_1 which are adjacent in $G(n, p)$ to more than $n^{1.01/5}$ vertices in R is at most $n^{2/5}$. We condition on the occurrence of this event and continue by exposing the edges in $G(n, p) \cap \binom{R}{2}$. For $M \subseteq G(n, p)$, let $E_1(M)$ be the set of edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq M$, which shares exactly three vertices with R and one vertex with $R_1 = [n] \setminus (R \cup R_0)$. Note that $|E_1(M)|$ is bounded by the number of paths of length two in $M \cap \binom{R}{2}$, whose three vertices are adjacent in $G(n, p)$ to a vertex in R_1 . Before the second exposure, the number of possible paths of this kind is at most

$$3 \cdot |R_1| \cdot (n^{1.01/5})^3 + 3 \cdot n^{2/5} \cdot (n^{2.02/5})^3 \leq n^{8.1/5}.$$

Hence, by Claim 6.2, with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, of these possible paths, only $O(n^{4.2/5})$ belong to $M \cap \binom{R}{2}$. So with probability at least $1 - n^{-\omega(s)}$, for every $M \subseteq G(n, p)$, assuming the maximum degree in $M \cap \binom{R}{2}$ is at most $n^{1.1/5}$, we have $|E_1(M)| = O(n^{4.2/5})$. This completes the proof.

Property (C8). Fix a set $R \subseteq [n]$ of r vertices, where $s - o(s) \leq r \leq s$, and a set E of $O(n^{1/2})$ edges in $\binom{[n]}{2} \setminus \binom{R}{2}$. To show that (C8) occurs with probability at least $1 - n^{-\omega(1)}$, it is enough to show that R and E satisfies the assertion of (C8) with probability at least $1 - n^{-\omega(s)}$. Furthermore, it suffices to do so conditioned on (C3). Let E_1 be the set of edges $g \in \binom{R}{2}$ for which there exists a graph $G \in X_{0,5}(g)$, with $G \subseteq G(n, p) \setminus \binom{R}{2}$ and $G \cap E \neq \emptyset$. Let E_2 be the number of paths of length two in $G(n, p) \cap \binom{R}{2}$ that complete some edge $g \in E_1$ to a triangle. Start by exposing only the edges in $G(n, p) \setminus \binom{R}{2}$. Conditioned on (C3), we have that $|E_1| \leq |E| \cdot n^{2/5+20\varepsilon_3} = O(n^{4.5/5+20\varepsilon_3})$. Continue by exposing the edges in $G(n, p) \cap \binom{R}{2}$. The number of possible paths of length two in $\binom{R}{2}$ that complete some edge $g \in E_1$ to a triangle is at most $|E_1| \cdot s = O(n^{7.6/5})$. Therefore, assuming the maximum degree in $G(n, p) \cap \binom{R}{2}$ is at most $n^{1.1/5}$, by Claim 6.2, with probability at least $1 - n^{-\omega(s)}$, $|E_2| = O(n^{7.6/5-3.99/5} + n^{4.2/5})$. This completes the proof. ■

We continue with the following conditional lower bound on the probability of \mathcal{D}_i .

Lemma 6.3. *Assume that M_i is given so that $\mathcal{A}_i \wedge \mathcal{B}_i$ holds. Further assume that the graph Traversed_i has the following properties: first, the maximum degree is at most $n^{3/5+10\epsilon_3}$; second, the number of vertices that are adjacent to any two fixed vertices is at most $n^{1/5+10\epsilon_3}$; third, the number of vertices that are adjacent to any three fixed vertices is at most $(\ln n)^{O(1)}$. Then the probability of \mathcal{D}_i (over the choice of BIGBite_{i+1}) is at least $1 - n^{-\omega(1)}$.*

Proof. We begin by noting that by Chernoff's bound, (D1) occurs with probability at least $1 - n^{-\omega(1)}$.

We continue by arguing that under the assumptions in the lemma, (D2) occurs with probability at least $1 - n^{-\omega(1)}$. Let $1 \leq j \leq 5$ and $f \in \text{NotTraversed}_i$. We assume that either $i \geq 1$, or else $j = 5$, since otherwise trivially $|X'_{i,j}(f)| = n^{(\epsilon_3-2/5)j}x_{i,j} = 0$ and we are done. Let $\{G'_l : l \in L\}$ be the set $X_{i,j}(f)$. Let $\{G_l : l \in L\}$ be the family (potentially a multiset) for which it holds that $G_l = G'_l \cap \text{NotTraversed}_i$ for every $l \in L$. Consider the binomial random graph $G(n, p)$ with $p = n^{\epsilon_3-2/5}$, and let W be as defined at the beginning of Section 2. Since \mathcal{A}_i holds and $|L| = |X_{i,j}(f)|$, we have $\mathbb{E}(W) = n^{(\epsilon_3-2/5)j}|L| \in n^{(\epsilon_3-2/5)j}x_{i,j}(1 \pm 1000\Gamma_i)$. Observe that $|X'_{i,j}(f)|$ has the same distribution as W and so, by the union bound, it is enough to prove that the probability that W deviates from its expectation by more than $1000\Gamma_i n^{(\epsilon_3-2/5)j}x_{i,j} \geq \mathbb{E}(W)^{0.9} = n^{\Omega(\epsilon_3)}$ is at most $n^{-\omega(1)}$. We do so using Theorem 2.1, with $\mathfrak{E}_0 = \mathbb{E}(W)$, $\mathfrak{E}_k = \exp((2j-k)\sqrt{\ln n})$ for $1 \leq k \leq j$, and $\lambda = (\ln n)^2$. We have several cases.

- Assume $j = 5$. If $3 \leq |G| \leq 5$ then trivially $|L_G| \leq 1$, while if $1 \leq |G| \leq 2$ then trivially $|L_G| \leq n$. Therefore, $\mathbb{E}_k(W) \leq 1$ for all $1 \leq k \leq 5$. Now note that $\mathbb{E}_0(W) = \mathbb{E}(W)$ and apply Theorem 2.1 with the above parameters.
- Assume $j = 4$. If $3 \leq |G| \leq 4$ then trivially $|L_G| \leq 1$. If $|G| = 2$ then since by assumption the maximum degree in the graph Traversed_i is at most $n^{3/5+10\epsilon_3}$, we have $|L_G| = O(n^{3/5+10\epsilon_3})$. If $|G| = 1$ then trivially $|L_G| \leq n$. Therefore, $\mathbb{E}_k(W) \leq 1$ for all $1 \leq k \leq 4$. As before, note that $\mathbb{E}_0(W) = \mathbb{E}(W)$ and apply Theorem 2.1 with the above parameters.
- Assume $j = 3$. If $|G| = 3$ then trivially $|L_G| \leq 1$. If $|G| = 2$ then since by assumption the number of vertices that are adjacent in Traversed_i to any two fixed vertices is at most $n^{1/5+10\epsilon_3}$, we have $|L_G| = O(n^{1/5+10\epsilon_3})$. If $|G| = 1$ then since by assumption the maximum degree in the graph Traversed_i is at most $n^{3/5+10\epsilon_3}$, we have $|L_G| = O(n^{3/5+10\epsilon_3})$. Therefore, $\mathbb{E}_k(W) \leq 1$ for all $1 \leq k \leq 3$. Apply Theorem 2.1 with the above parameters.
- Assume $j = 2$. If $|G| = 2$ then since by assumption the number of vertices that are adjacent in Traversed_i to any three fixed vertices is at most $(\ln n)^{O(1)}$, we have $|L_G| \leq (\ln n)^{O(1)}$. If $|G| = 1$ then since by assumption the number of vertices that are adjacent in Traversed_i to any two fixed vertices is at most $n^{1/5+10\epsilon_3}$, we have $|L_G| = O(n^{1/5+10\epsilon_3})$. Therefore, $\mathbb{E}_k(W) \leq (\ln n)^{O(1)}$ for all $1 \leq k \leq 2$. Apply Theorem 2.1 with the above parameters.
- Assume $j = 1$. If $|G| = 1$ then since by assumption the number of vertices that are adjacent in Traversed_i to any three fixed vertices is at most $(\ln n)^{O(1)}$, we have $|L_G| \leq (\ln n)^{O(1)}$. Therefore, $\mathbb{E}_1(W) \leq (\ln n)^{O(1)}$. Apply Theorem 2.1 with the above parameters.

We end by arguing that under the assumptions in the lemma, (D3) occurs with probability at least $1 - n^{-\omega(1)}$. Let $S \subseteq [n]$ be a set of s vertices, let $S_i \subseteq S$ be the set that is guaranteed to exist by \mathcal{B}_i , let $1 \leq k < j \leq 3$, let $(R, T) \in \text{Pairs}(S_i)$ and let $t = |T|$, noting that $t = \Omega(s^3)$. Note that by (B2), $\text{Traversed}_i \cap \binom{R}{2}$ has maximum degree at most $n^{1.1/5}$, and that by (B3), $|Y_{i,j}(T)| \geq y_{i,j,t}(1 - 100\Gamma_i)$ (as $2 \leq j \leq 3$). By the union bound, it is enough to show that $|Y'_{i,j,k}(T)| \geq n^{(\varepsilon_3 - 2/5)k} \binom{j}{k} y_{i,j,t}(1 - 100\Gamma_i - \Gamma_i \gamma_i)$ occurs with probability at least $1 - n^{-\omega(s)}$. Let $\{G_l : l \in L\}$ be the multiset $\biguplus_{(G_1, G_2, G_3)} G_2$, where the multiset union ranges over all $(G_1, G_2, G_3) \in Y_{i,j,k}(T)$. Consider the binomial random graph $G(n, p)$ with $p = n^{\varepsilon_3 - 2/5}$, and let W be as defined at the beginning of Section 2. By assumption, $|L| = \binom{j}{k} |Y_{i,j}(T)| \geq \binom{j}{k} y_{i,j,t}(1 - 100\Gamma_i)$ and so $\mathbb{E}(W) \geq n^{(\varepsilon_3 - 2/5)k} \binom{j}{k} y_{i,j,t}(1 - 100\Gamma_i)$. Since $|Y'_{i,j,k}(T)|$ has the same distribution as W , it remains for us to show that the probability that $W < \mathbb{E}(W) - \lambda$ for $\lambda = \Gamma_i \gamma_i n^{(\varepsilon_3 - 2/5)k} \binom{j}{k} y_{i,j,t}$, is at most $n^{-\omega(s)}$. We do so using Theorem 2.2, and for that it is enough to show that $\lambda^2 / (\mathbb{E}(W) + \Delta) = \omega(s \ln n)$, where Δ is as defined in Section 2. We have three cases.

- Assume that $k = 2$ and $j = 3$. Clearly, $\Delta \leq 12s^4 n^{3(\varepsilon_3 - 2/5)} \leq n^{6.1/5}$. Furthermore, $\mathbb{E}(W) \leq 3s^3 n^{2(\varepsilon_3 - 2/5)} \leq n^{5.1/5}$. Also, $\lambda \geq n^{5/5}$ by Lemma 3.1. Thus, $\lambda^2 / (\mathbb{E}(W) + \Delta) = \omega(s \ln n)$.
- Assume that $k = 1$ and $j = 3$. Clearly, $\Delta \leq 3s^4 n^{\varepsilon_3 - 2/5} \leq n^{10.1/5}$. Furthermore, $\mathbb{E}(W) \leq 3s^3 n^{\varepsilon_3 - 2/5} \leq n^{7.1/5}$. Also, $\lambda \geq n^{7/5}$ by Lemma 3.1. Thus, $\lambda^2 / (\mathbb{E}(W) + \Delta) = \omega(s \ln n)$.
- Assume that $k = 1$ and $j = 2$. Recall that $|L| \geq \binom{j}{k} y_{i,j,t}(1 - 100\Gamma_i)$. It is safe to assume that $|L| \leq \binom{j}{k} y_{i,j,t}$ (since otherwise we can remove some of the members of L so that this assumption does hold; such an alteration will not affect the proof). By this upper bound on $|L|$, and using the fact that the maximum degree in $\text{Traversed}_i \cap \binom{R}{2}$ is at most $n^{1.1/5}$, one can verify using Lemma 3.1 that $\Delta \leq 2|L| n^{1.1/5} n^{\varepsilon_3 - 2/5} \leq n^{6.2/5}$. Furthermore, by this upper bound on $|L|$ and by Lemma 3.1, $\mathbb{E}(W) = |L| n^{\varepsilon_3 - 2/5} \leq n^{5.1/5}$. Also, $\lambda \geq n^{5/5}$ by Lemma 3.1. Thus, $\lambda^2 / (\mathbb{E}(W) + \Delta) = \omega(s \ln n)$. ■

7 Proof of Lemma 3.2

In this section we prove our main lemma. We start with the following lemma.

Lemma 7.1. *For $0 \leq i < I$,*

$$\Pr(\mathcal{A}_i \wedge \mathcal{B}_i) \geq 1 - in^{-0.1} \implies \Pr(\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq 1 - in^{-0.1} - n^{-\omega(1)}.$$

Proof. Let $0 \leq i < I$. Assume that $\Pr(\mathcal{A}_i \wedge \mathcal{B}_i) \geq 1 - in^{-0.1}$. Standard arguments show that with probability at least $1 - n^{-\omega(1)}$, the graph Traversed_i satisfies the three properties that are assumed to hold in Lemma 6.3. Therefore, by Lemma 6.3 we have $\Pr(\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{D}_i) \geq 1 - in^{-0.1} - n^{-\omega(1)}$. The lemma now follows since by Lemma 6.1 we have $\Pr(\mathcal{C}_i) \geq 1 - n^{-\omega(1)}$. ■

The proof of Lemma 3.2 is by induction on i . Trivially, $\Pr(\mathcal{A}_0 \wedge \mathcal{B}_0) = 1$. This, together with Lemma 7.1, implies the validity of the lemma for the case $i = 0$. Assume the lemma holds for

$0 \leq i < I - 1$. We prove that $\Pr(\mathcal{A}_{i+1} \wedge \mathcal{B}_{i+1} \wedge \mathcal{C}_{i+1} \wedge \mathcal{D}_{i+1}) \geq 1 - (i+1)n^{-0.1} - n^{-\omega(1)}$. To do that, by Lemma 7.1 it suffices to prove that $\Pr(\mathcal{A}_{i+1} \wedge \mathcal{B}_{i+1}) \geq 1 - (i+1)n^{-0.1}$. Thus, by the induction hypothesis, it is enough to prove that

$$\Pr(\mathcal{A}_{i+1} \mid \mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq 1 - n^{-0.11} \quad \text{and} \quad (11)$$

$$\Pr(\mathcal{B}_{i+1} \mid \mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq 1 - n^{-0.11}. \quad (12)$$

The proof of (11) and (12) is given in the next two subsections. In our arguments below we make use of the notion of an outcome of an edge in $BIGBite_{i+1}$: an outcome of such an edge is either the event that the edge is not in $Bite_{i+1}$, or otherwise it is the birthtime of the edge. Changing the outcome of an edge that is not in $Bite_{i+1}$ means adding that edge to $Bite_{i+1}$ and giving it an arbitrary birthtime. Changing the outcome of an edge in $Bite_{i+1}$ means either taking that edge out of $Bite_{i+1}$, or changing its birthtime arbitrarily.

We will also need the following definitions and observations. Given $BIGBite_{i+1}$, say that an edge $f \in BIGBite_{i+1}$ has the potential of being a label in a tree $\mathfrak{T}_c(g)$ or in a tree $\mathfrak{T}_c(g, S)$, where $g \in NotTraversed_i$ and $S \subseteq [n]$, if there exists a choice of $BigBite_{i+1} \subseteq BIGBite_{i+1}$ so that given that particular choice of $BigBite_{i+1}$, indeed f is a label in $\mathfrak{T}_c(g)$ or $\mathfrak{T}_c(g, S)$, respectively. (Recall that $BigBite_{i+1}$ completely determines $\mathfrak{T}_c(g)$ and $\mathfrak{T}_c(g, S)$, and that those labels of $\mathfrak{T}_c(g)$ and $\mathfrak{T}_c(g, S)$ that are edges, are all in $BigBite_{i+1}$, except maybe for the label of the root.) The motivation behind these definitions is that, for example, whenever $f \in BIGBite_{i+1}$ has the potential of being a label in a tree $\mathfrak{T}_c(g)$, then we know that the outcome of f could affect the occurrence of $\mathcal{S}_c(g)$; otherwise, the outcome of f would not affect the occurrence of $\mathcal{S}_c(g)$. For an edge $f \in NotTraversed_i$, let $Labels_c(f)$ be the set of edges in $BIGBite_{i+1}$ which have the potential of being labels in $\mathfrak{T}_c(f)$. For an edge $f \in BIGBite_{i+1}$, let $Roots_c(f)$ be set of edges $g \in NotTraversed_i$ such that f has the potential of being a label in $\mathfrak{T}_c(g)$. Assuming $\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i$ and $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$, it follows from (D2), (C5) and from our choice of ε_2 that

$$f \in NotTraversed_i \implies |Labels_c(f)| \leq 1 + 5^c \cdot n^{10\varepsilon_3 c} \leq n^{0.01}, \quad (13)$$

$$f \in BIGBite_{i+1} \implies |Roots_c(f) \cap BIGBite_{i+1}| \leq 1 + 5^c \cdot n^{10\varepsilon_3 c} \leq n^{0.01} \quad \text{and} \quad (14)$$

$$f \in BIGBite_{i+1} \implies |Roots_c(f)| \leq 1 + 5^c \cdot n^{10\varepsilon_3 c} \cdot n^{2/5+10\varepsilon_3} \leq n^{2.1/5}. \quad (15)$$

7.1 Proof of (11)

Let \mathcal{E}_4 and \mathcal{E}_5 be, respectively, the events

$$|M_{i+1}| \in 0.5n^{8/5}\Phi((i+1)n^{-\varepsilon_1})(1 \pm 100\Gamma_{i+1}) \quad \text{and}$$

$$|O_{i+1}| \in 0.5n^2\phi((i+1)n^{-\varepsilon_1})(1 \pm 100\Gamma_{i+1}).$$

Lemma 7.2. *Assume that M_i and $BIGBite_{i+1}$ are given so that $\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i$ holds. Then the probability of $\mathcal{E}_4 \wedge \mathcal{E}_5$ (over the choice of $BigBite_{i+1}$, $Bite_{i+1}$ and the choice of the birthtimes of the edges in $Bite_{i+1}$) is at least $1 - n^{-\omega(1)}$.*

Proof. Fix an integer $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$. Let \sum_f range over all $f \in O_i$. Let

$$W_1 := \sum_f \mathbf{1}[\mathcal{S}_c(f) \wedge f \in Bite_{i+1}] \quad \text{and} \quad W_2 := \sum_f \mathbf{1}[\mathcal{S}_c(f) \wedge f \notin Bite_{i+1}].$$

By Lemma 5.1, we have that if c is odd then $|M_{i+1} \setminus M_i| \geq W_1$ and $|O_{i+1}| \geq W_2$, while if c is even then $|M_{i+1} \setminus M_i| \leq W_1$ and $|O_{i+1}| \leq W_2$. Now, (A1) and the fact that $\Gamma_i \leq \Gamma_{i+1}$ imply that $|M_i| \in 0.5n^{8/5}\Phi(in^{-\varepsilon_1})(1 \pm 100\Gamma_{i+1})$. So it suffices to prove that $W_1 \in 0.5n^{8/5}(\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1}))(1 \pm 100\Gamma_{i+1})$ and $W_2 \in 0.5n^2\phi((i+1)n^{-\varepsilon_1})(1 \pm 100\Gamma_{i+1})$, each occurs with probability at least $1 - n^{-\omega(1)}$.

Since by (A2) the number of edges over which \sum_f ranges is in $0.5n^2\phi(in^{-\varepsilon_1})(1 \pm 100\Gamma_i)$, since by (D1) the number of edges in $BIGBite_{i+1}$ over which \sum_f ranges is in $0.5n^{8/5+\varepsilon_3}\phi(in^{-\varepsilon_1})(1 \pm (100\Gamma_i + \Gamma_i\gamma_i))$, and since $\mathcal{C}_i \wedge \mathcal{D}_i$ holds, we may apply Lemma 5.4 to get

$$\begin{aligned}\mathbb{E}(W_1) &\in 0.5n^{8/5}(\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1}))(1 \pm (100\Gamma_i + 99\Gamma_i\gamma_i)) \quad \text{and} \\ \mathbb{E}(W_2) &\in 0.5n^2\phi((i+1)n^{-\varepsilon_1})(1 \pm (100\Gamma_i + 99\Gamma_i\gamma_i)).\end{aligned}$$

Observe that the above estimate on $\mathbb{E}(W_2)$, together with Lemma 3.1, implies that $\mathbb{E}(W_2) \geq n^{9.9/5}$. Also, the above estimate on $\mathbb{E}(W_1)$, together with Lemma 3.1, and the fact that $\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1}) \geq n^{-\Theta(\varepsilon_1)}$ (indeed, recall the proof of Lemma 5.4, where we have argued indirectly that $(\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})) / (n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})) = \Omega(1)$), implies that $\mathbb{E}(W_1) \geq n^{7.9/5}$. Therefore, by Lemma 3.1, it suffices to show that the probability that W_1 and W_2 deviate from their expectation by more than $n^{7/5}$ is at most $n^{-\omega(1)}$.

Note that W_1 and W_2 each depends only on the outcomes of edges in $BIGBite_{i+1}$, which by (C1) contains at most $n^{8.1/5}$ edges. Furthermore, by (15), every edge in $BIGBite_{i+1}$ has the potential of being a label in at most $n^{2.1/5}$ trees $\mathfrak{T}_c(f)$ with $f \in O_i$. This implies that changing the outcome of a single edge in $BIGBite_{i+1}$ can change W_1 and W_2 each by at most an additive factor of $n^{2.1/5}$. Therefore, by McDiarmid's inequality we can conclude that the probability that W_1 or W_2 each deviates from its expectation by more than $n^{7/5}$ is at most $n^{-\omega(1)}$. \blacksquare

From Lemma 7.2 it follows that

$$\Pr(\mathcal{E}_4 \wedge \mathcal{E}_5 \mid \mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq 1 - n^{-\omega(1)}.$$

Let \mathcal{E}_6 be the following event: letting $m_{i+1} := |M_{i+1}|$, for all $1 \leq j \leq 5$ and all $f \in \text{NotTraversed}_i$,

$$\begin{aligned}|O_{i+1}| &\in 0.5n^2 \exp(-16(m_{i+1}n^{-8/5})^5)(1 \pm n^{-\varepsilon_3}) \quad \text{and} \\ |X_{i+1,j}(f)| &\in n^{2j/5}2^{4-j}\binom{5}{j}(m_{i+1}n^{-8/5})^{5-j} \exp(-16j(m_{i+1}n^{-8/5})^5)(1 \pm n^{-\varepsilon_3}).\end{aligned}$$

A result of Bohman [2, Theorem 13] implies that $\Pr(\mathcal{E}_6) \geq 1 - n^{-1/6}$, and so by the induction hypothesis,

$$\Pr(\mathcal{E}_6 \mid \mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) \geq \frac{\Pr(\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i) - n^{-1/6}}{\Pr(\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i)} \geq 1 - 2n^{-1/6}.$$

Since our goal is to prove (11), it remains to argue that $\mathcal{E}_4 \wedge \mathcal{E}_5 \wedge \mathcal{E}_6$ imply the bounds on $|X_{i+1,j}(f)|$ that are asserted by \mathcal{A}_{i+1} , for all $1 \leq j \leq 5$ and all $f \in \text{NotTraversed}_i$. Indeed, note that $\mathcal{E}_4 \wedge \mathcal{E}_5 \wedge \mathcal{E}_6$ implies

$$\begin{aligned}m_{i+1} &\in 0.5n^{8/5}\Phi((i+1)n^{-\varepsilon_1})(1 \pm 100\Gamma_{i+1}) \quad \text{and} \\ \exp(-16(m_{i+1}n^{-8/5})^5) &\in \phi((i+1)n^{-\varepsilon_1})(1 \pm 101\Gamma_{i+1}).\end{aligned}$$

This in turn implies that for all $1 \leq j \leq 5$,

$$\begin{aligned} 2^{4-j}(m_{i+1}n^{-8/5})^{5-j} \exp(-16j(m_{i+1}n^{-8/5})^5)(1 \pm n^{-\varepsilon_3}) &\subseteq \\ 0.5(\Phi((i+1)n^{-\varepsilon_1}))^{5-j} \phi((i+1)n^{-\varepsilon_1})^j (1 \pm 999\Gamma_{i+1}). \end{aligned}$$

This, together with the fact that $0.5n^{2j/5} \in \binom{n}{2} \left(\frac{1}{n^{2/5}}\right)^{5-j} (1 \pm o(\Gamma_{i+1}))$, completes the proof.

7.2 Proof of (12)

Fix for the rest of the section a set $S \subseteq [n]$ of s vertices. Further, assume that we are given M_i and $BIGBite_{i+1}$ so that $\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{D}_i$ holds. Under this assumption, we prove that with probability at least $1 - n^{-\omega(s)}$ (where the probability is over the choice of $BigBite_{i+1}$, $Bite_{i+1}$ and the choice of the birthtimes of the edges in $Bite_{i+1}$), there exists a set $S_{i+1} \subseteq S$, which satisfies the three properties that are asserted to hold by \mathcal{B}_{i+1} . A union bound argument will then give us (12).

We start by defining the set S_{i+1} . Let $S_i \subseteq S$ be the set that is guaranteed to exist by \mathcal{B}_i . Let S_{i+1} be the set of all vertices in S_i whose degree in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{S_i}{2}$ is at most $n^{1.1/5}$, so that

$$\text{the maximum degree in } (Traversed_i \cup BIGBite_{i+1}) \cap \binom{S_{i+1}}{2} \text{ is at most } n^{1.1/5}. \quad (16)$$

We claim that with probability 1, S_{i+1} satisfies the first two properties that are asserted to hold by \mathcal{B}_{i+1} . Indeed, by assumption, S_i has size at least $s(1 - in^{-0.01}) = \omega(n^{3/5})$. This, together with (C2), implies that the average degree in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{S_i}{2}$ is at most $n^{1/5+10\varepsilon_3}$. Hence, the number of vertices in S_i whose degree in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{S_i}{2}$ is more than $n^{1.1/5}$ is at most $|S_i|n^{-0.01} \leq sn^{-0.01}$. It follows that S_{i+1} has size at least $|S_i| - sn^{-0.01} \geq s(1 - (i+1)n^{-0.01})$. In addition to that, from the fact that $Traversed_{i+1} \subseteq Traversed_i \cup BIGBite_{i+1}$ and from (16) it follows that $Traversed_{i+1} \cap \binom{S_{i+1}}{2}$ has maximum degree at most $n^{1.1/5}$.

It is left for us to show that with the desired probability, S_{i+1} satisfies the third property that is asserted to hold by \mathcal{B}_{i+1} . Fix for the rest of the section a pair $(R, T) \in Pairs(S_{i+1})$ and let $t = |T|$, noting that $t = \Omega(s^3)$. By the union bound, it remains to argue that for every $1 \leq j \leq 3$, with probability at least $1 - n^{-\omega(s)}$, R and T satisfy $\mathcal{B}_{i+1}(\text{B3})$. That is, it remains to prove that for every $1 \leq j \leq 3$,

$$\Pr(|Y_{i+1,j}(T)| \geq y_{i+1,j,t}(1 - 100\Gamma_{i+1}) - 0.5j(3-j)(2-j)|Z_{i+1}(R, T)|) \geq 1 - n^{-\omega(s)}, \quad (17)$$

where we stress again that the probability is over the choice of $BigBite_{i+1}$, $Bite_{i+1}$, and the choice of the birthtimes of the edges in $Bite_{i+1}$.

In order to prove (17), we assume that the inequalities

$$|Y_{i,j}(T)| \leq y_{i,j,t} \quad \text{and} \quad |Y'_{i,j,k}(T)| \leq n^{(\varepsilon_3 - 2/5)k} \binom{j}{k} y_{i,j,t} \quad (18)$$

hold for every possible choice of j and k . This is a safe assumption since by (B3) and (D3) we can always remove, if needed, elements from these sets so that this assumption holds. Such an alteration will not affect the proof.

7.2.1 Proof of (17) (case $j = 3$)

Let $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$ be an odd integer and let

$$W_3 := \sum_{G \in Y_{i,3}(T)} \mathbf{1}[\mathcal{I}_c(G) \wedge \mathcal{S}_c(G) \wedge |G \cap \text{Bite}_{i+1}| = 0].$$

By Lemma 5.1, we have $|Y_{i+1,3}(T)| \geq W_3$. This, with the next two claims, gives (17) for $j = 3$.

Claim 7.3. $\mathbb{E}(W_3) \geq y_{i+1,3,t}(1 - 100\Gamma_i - 99\Gamma_i\gamma_i)$.

Proof. Consider a triangle $G \in Y_{i,3}(T)$. Since an edge in BIGBite_{i+1} is an edge in Bite_{i+1} with probability $n^{-\varepsilon_1 - \varepsilon_3} / (1 - in^{-\varepsilon_1 - \varepsilon_2})$, using Lemma 3.1, we find that $\Pr(|G \cap \text{Bite}_{i+1}| = 0) \geq 1 - o(\Gamma_i\gamma_i)$. Hence, by Lemma 5.4,

$$\Pr(\mathcal{I}_c(G) \wedge \mathcal{S}_c(G) \wedge |G \cap \text{Bite}_{i+1}| = 0) \geq \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^3 (1 - 91\Gamma_i\gamma_i).$$

Also, by (B3), $|Y_{i,3}(T)| \geq y_{i,3,t}(1 - 100\Gamma_i)$. The claim now follows using linearity of expectation. \blacksquare

Claim 7.4. *The probability (over the choice of BigBite_{i+1} , Bite_{i+1} , and the choice of the birthtimes of the edges in Bite_{i+1}) that W_3 deviates from its expectation by more than $\Gamma_i\gamma_i y_{i+1,3,t}$ is at most $n^{-\omega(s)}$.*

Proof. For an edge $f \in \text{BIGBite}_{i+1}$, let $\text{Triangles}(f)$ be the set which contains every triangle $G \in Y_{i,3}(T)$ for which it holds that f has the potential of being a label in a tree in the forest $\{\mathcal{T}_c(g) : g \in G\}$. Observe that changing the outcome of an edge $f \in \text{BIGBite}_{i+1}$ can change W_3 by at most an additive factor of $|\text{Triangles}(f)|$.

By (15) we have that for every $f \in \text{BIGBite}_{i+1}$, $|\text{Triangles}(f)| \leq |\text{Roots}_c(f)| \cdot s \leq n^{5.2/5}$. By (13) we have that for every triangle $G \in Y_{i,3}(T)$, there are at most $3n^{0.01}$ edges $f \in \text{BIGBite}_{i+1}$ such that $G \in \text{Triangles}(f)$, and so

$$\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)| \leq |Y_{i,3}(T)| \cdot 3n^{0.01} \leq n^{9.1/5},$$

where the second inequality follows since trivially $|Y_{i,3}(T)| \leq s^3$. Therefore,

$$\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)|^2 \leq n^{5.2/5} \cdot \sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)| \leq n^{14.3/5}.$$

It now follows from McDiarmid's inequality that the probability that W_3 deviates from its expectation by more than $\Gamma_i\gamma_i y_{i+1,3,t} \geq n^{8.9/5}$ is at most $n^{-\omega(s)}$. \blacksquare

7.2.2 Proof of (17) (case $j = 2$)

Let $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$ be an odd integer. Let \sum_G range over all triangles $G \in Y_{i,2}(T)$ and let $\sum_{(G_1, G_2, G_3)}$ range over all triples $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$. Let

$$\begin{aligned} W_4 := & \sum_G \mathbf{1}[\mathcal{I}_c(G \cap \text{NotTraversed}_i) \wedge \mathcal{S}_c(G \cap \text{NotTraversed}_i) \wedge |G \cap \text{Bite}_{i+1}| = 0] + \\ & \sum_{(G_1, G_2, G_3)} \mathbf{1}[\mathcal{I}_c(G_2 \cup G_3) \wedge \mathcal{S}_c(G_2 \cup G_3) \wedge G_2 \subseteq \text{Bite}_{i+1} \wedge |G_3 \cap \text{Bite}_{i+1}| = 0]. \end{aligned}$$

By Lemma 5.1, we have $|Y_{i+1,2}(T)| \geq W_4$. This, with the next two claims, gives (17) for $j = 2$.

Claim 7.5. $\mathbb{E}(W_4) \geq y_{i+1,2,t}(1 - 100\Gamma_i - 99\Gamma_i\gamma_i)$.

Proof. Consider a triangle $G \in Y_{i,2}(T)$ and a triple $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$. Since an edge in $BIGBite_{i+1}$ is an edge in $Bite_{i+1}$ with probability $n^{-\varepsilon_1-\varepsilon_3}/(1 - in^{-\varepsilon_1-\varepsilon_2})$, using Lemma 3.1, we find that

$$\begin{aligned} \Pr(|G \cap Bite_{i+1}| = 0) &\geq 1 - o(\Gamma_i\gamma_i) \quad \text{and} \\ \Pr(G_2 \subseteq Bite_{i+1} \wedge |G_3 \cap Bite_{i+1}| = 0) &\geq n^{-\varepsilon_1-\varepsilon_3}(1 - o(\Gamma_i\gamma_i)). \end{aligned}$$

Hence, by Lemma 5.4,

$$\begin{aligned} \Pr(\mathcal{I}_c(G \cap NotTraversed_i) \wedge \mathcal{S}_c(G \cap NotTraversed_i) \wedge |G \cap Bite_{i+1}| = 0) &\geq \\ &\left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^2 (1 - 91\Gamma_i\gamma_i) \end{aligned}$$

and

$$\begin{aligned} \Pr(\mathcal{I}_c(G_2 \cup G_3) \wedge \mathcal{S}_c(G_2 \cup G_3) \wedge G_2 \subseteq Bite_{i+1} \wedge |G_3 \cap Bite_{i+1}| = 0) &\geq \\ n^{-\varepsilon_1-\varepsilon_3} \left(\frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})} \right) \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^2 (1 - 91\Gamma_i\gamma_i). \end{aligned}$$

Also, by (B3), $|Y_{i,2}(T)| \geq y_{i,2,t}(1 - 100\Gamma_i)$, and by (D3), $|Y'_{i,3,1}(T)| \geq 3n^{\varepsilon_3-2/5}y_{i,3,t}(1 - 100\Gamma_i - \Gamma_i\gamma_i)$. The claim now follows using linearity of expectation. \blacksquare

Claim 7.6. *The probability (over the choice of $BIGBite_{i+1}$, $Bite_{i+1}$, and the choice of the birthtimes of the edges in $Bite_{i+1}$) that W_4 deviates from its expectation by more than $\Gamma_i\gamma_i y_{i+1,2,t}$ is at most $n^{-\omega(s)}$.*

Proof. For an edge $f \in BIGBite_{i+1}$, let $Triangles(f)$ be the set which contains every triangle $G \in Y_{i,2}(T)$ for which it holds that f has the potential of being a label in a tree in the forest $\{\mathfrak{T}_c(g) : g \in G \cap NotTraversed_i\}$, and every triple $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$ for which it holds that f has the potential of being a label in a tree in the forest $\{\mathfrak{T}_c(g) : g \in G_2 \cup G_3\}$. Observe that changing the outcome of an edge $f \in BIGBite_{i+1}$ can change W_4 by at most an additive factor of $|Triangles(f)|$.

Note that every triangle $G \in Y_{i,2}(T)$ has at least one edge in $Traversed_i \cup BIGBite_{i+1}$, and that every triple $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$ is such that the triangle $G_1 \cup G_2 \cup G_3$ has at least one edge in $Traversed_i \cup BIGBite_{i+1}$. This fact, together with (14), (15), (16) and the fact that for every triangle G there are at most 3 possible triples $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$ such that $G = G_1 \cup G_2 \cup G_3$, gives us that for every $f \in BIGBite_{i+1}$,

$$|Triangles(f)| \leq 3 \cdot |Roots_c(f)| \cdot 2n^{1.1/5} + 3 \cdot |Roots_c(f) \cap BIGBite_{i+1}| \cdot s \leq 10n^{3.2/5}.$$

Moreover, by (13) we have that for every triangle $G \in Y_{i,2}(T)$, there are at most $2n^{0.01}$ edges $f \in BIGBite_{i+1}$ such that $G \in Triangles(f)$, and likewise for every triple $(G_1, G_2, G_3) \in Y'_{i,3,1}(T)$, there are at most $3n^{0.01}$ edges $f \in BIGBite_{i+1}$ such that $(G_1, G_2, G_3) \in Triangles(f)$, and so

$$\sum_{f \in BIGBite_{i+1}} |Triangles(f)| \leq (|Y_{i,2}(T)| + |Y'_{i,3,1}(T)|) \cdot 3n^{0.01} \leq n^{7.1/5},$$

where the second inequality follows from (18). Therefore,

$$\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)|^2 \leq 10n^{3.2/5} \cdot \sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)| \leq 10n^{10.3/5}.$$

It now follows from McDiarmid's inequality that the probability that W_4 deviates from its expectation by more than $\Gamma_i \gamma_i y_{i+1,2,t} \geq n^{6.9/5}$ is at most $n^{-\omega(s)}$. \blacksquare

7.2.3 Proof of (17) (case $j = 1$)

We begin with the following claim.

Claim 7.7. *Let $Y_1(T) := Y_{i,1}(T) \cup Y'_{i,2,1}(T) \cup Y'_{i,3,2}(T)$ be a set of triangles and triples. There exists a set $Y(T) \subseteq Y_1(T)$, of size at least $|Y_1(T)| - O(n^{4.2/5})$, such that for every edge g_1 there is at most one triangle $\{g_1, g_2, g_3\}$ with $\{g_2, g_3\} \subseteq \text{Traversed}_i \cup \text{BIGBite}_{i+1}$, such that either $\{g_1, g_2, g_3\} \in Y(T)$, or $\{g_1, g_2, g_3\} = G_1 \cup G_2 \cup G_3$ for some triple $(G_1, G_2, G_3) \in Y(T)$.*

Proof. By (C6), there is a set E_0 of at most $n^{3/5+10\varepsilon_3}$ edges, the removal of which from $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ leaves at most $n^{4/5+10\varepsilon_3}$ 4-cycles in $(\text{Traversed}_i \cup \text{BIGBite}_{i+1}) \cap \binom{R}{2}$. Obtain $Y_2(T)$ from $Y_1(T)$ by removing from $Y_1(T)$ every triangle G for which it holds that $G \cap E_0 \neq \emptyset$, and every triple (G_1, G_2, G_3) for which it holds that $(G_1 \cup G_2 \cup G_3) \cap E_0 \neq \emptyset$. Note that every triangle in $Y_1(T)$ has at least two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$, and that every triple (G_1, G_2, G_3) in $Y_1(T)$ is such that the triangle $G_1 \cup G_2 \cup G_3$ has at least two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$. Hence, using (16), we find that every edge in E_0 belongs to at most $2n^{1.1/5}$ triangles in $Y_1(T)$, and to at most $2n^{1.1/5}$ triples $G_1 \cup G_2 \cup G_3$ such that $(G_1, G_2, G_3) \in Y_1(T)$. Since for every triangle G there are at most 3 triples $(G_1, G_2, G_3) \in Y_1(T)$ such that $G = G_1 \cup G_2 \cup G_3$, we get that $|Y_2(T)| = |Y_1(T)| - |E_0| \cdot O(n^{1.1/5})$, and so $|Y_2(T)| = |Y_1(T)| - O(n^{4.2/5})$.

Say that a triangle $G \in Y_2(T)$ is bad if it has two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ that belong to some 4-cycle in $(\text{Traversed}_i \cup \text{BIGBite}_{i+1}) \cap \binom{R}{2}$. Say that a triple $(G_1, G_2, G_3) \in Y_2(T)$ is bad if the triangle $G_1 \cup G_2 \cup G_3$ has two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ that belong to some 4-cycle in $(\text{Traversed}_i \cup \text{BIGBite}_{i+1}) \cap \binom{R}{2}$. Obtain $Y(T)$ from $Y_2(T)$ by removing from $Y_2(T)$ every bad triangle and every bad triple. Observe that $Y(T)$ satisfies the property that is asserted to hold by the claim, and so it remains for us to show that $|Y(T)| = |Y_1(T)| - O(n^{4.2/5})$. To show this, first recall that if we remove from $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ the edges in E_0 , the number of 4-cycles remaining in $(\text{Traversed}_i \cup \text{BIGBite}_{i+1}) \cap \binom{R}{2}$ is at most $n^{4/5+10\varepsilon_3} \leq n^{4.1/5}$. Then, note that for every 4-cycle there are exactly 4 triangles that share two of their edges with that cycle. It follows that the number of bad triangles and bad triples in $Y_2(T)$ is at most $O(n^{4.1/5})$. Thus, $|Y(T)| = |Y_2(T)| - O(n^{4.1/5})$, and so $|Y(T)| = |Y_1(T)| - O(n^{4.2/5})$. \blacksquare

Fix for the rest of the section the set $Y(T)$ that is guaranteed to exist by Claim 7.7. For brevity, we mark a few properties of $Y(T)$ for future reference. Let (P1) be the property that every triangle $G \in Y(T)$ has at least two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$ and that every triple $(G_1, G_2, G_3) \in Y(T)$ is such that the triangle $G_1 \cup G_2 \cup G_3$ has at least two edges in $\text{Traversed}_i \cup \text{BIGBite}_{i+1}$. Let (P2) be the property that for every triangle G there are at most 3 possible triples $(G_1, G_2, G_3) \in Y(T)$ such that $G = G_1 \cup G_2 \cup G_3$. Let (P3) be the property that every edge in $\text{NotTraversed}_i \setminus \text{BIGBite}_{i+1}$

belongs to at most 3 triangles and triples in $Y(T)$ (where we say that an edge belongs to a triple (G_1, G_2, G_3) if that edge belongs to $G_1 \cup G_2 \cup G_3$). Using the fact that $Y(T) \subseteq Y_1(T)$, where $Y_1(T)$ is as defined in Claim 7.7, and using Claim 7.7, we find that the properties (P1), (P2) and (P3) hold.

Let $c \in \varepsilon_3^2 \varepsilon_2^{-1} \pm 1$ be an odd integer. Let \sum_G range over all triangles $G \in Y(T)$ and let $\sum_{(G_1, G_2, G_3)}$ range over all triples $(G_1, G_2, G_3) \in Y(T)$. Let

$$W_5 := \sum_G \mathbf{1}[\mathcal{I}_c(\emptyset, G \cap \text{NotTraversed}_i, R) \wedge \mathcal{S}_c(\emptyset, G \cap \text{NotTraversed}_i, R) \wedge |G \cap \text{Bite}_{i+1}| = 0] + \sum_{(G_1, G_2, G_3)} \mathbf{1}[\mathcal{I}_c(G_2, G_3, R) \wedge \mathcal{S}_c(G_2, G_3, R) \wedge G_2 \subseteq \text{Bite}_{i+1} \wedge |G_3 \cap \text{Bite}_{i+1}| = 0].$$

By Lemma 5.3, there are at least W_5 triangles G such that $|M_{i+1} \cap G| = 2$, $|\text{NotTraversed}_{i+1} \cap G| = 1$, and letting g denote the edge in $\text{NotTraversed}_{i+1} \cap G$, one of the following two possibilities hold. Either $|X_{i+1,0}(g)| = 0$, in which case $G \in Y_{i+1,1}(T)$, or $|X_{i+1,0}(g)| > 0$ and for every $G_0 \in X_{i+1,0}(g)$, G_0 shares at least three vertices with R . The number of triangles G for which the second possibility holds is at most $|Z_{i+1}(R, T)| - |Z_i(R, T)|$. Therefore, there are at least $W_5 - |Z_{i+1}(R, T)| + |Z_i(R, T)|$ triangles G for which the first possibility holds, or in other words, $|Y_{i+1,1}(T)| \geq W_5 - |Z_{i+1}(R, T)| + |Z_i(R, T)|$. This, with the next two claims, gives (17) for $j = 1$.

Claim 7.8. $\mathbb{E}(W_5) \geq y_{i+1,1,t}(1 - 100\Gamma_i - 99\Gamma_i\gamma_i) - |Z_i(R, T)|$.

Proof. Consider a triangle $G \in Y(T)$ and a triple $(G_1, G_2, G_3) \in Y(T)$. Since an edge in BIGBite_{i+1} is an edge in Bite_{i+1} with probability $n^{-\varepsilon_1 - \varepsilon_3} / (1 - in^{-\varepsilon_1 - \varepsilon_2})$, using Lemma 3.1, we find that $\Pr(|G \cap \text{Bite}_{i+1}| = 0) \geq 1 - o(\Gamma_i\gamma_i)$ and $\Pr(G_2 \subseteq \text{Bite}_{i+1} \wedge |G_3 \cap \text{Bite}_{i+1}| = 0) \geq n^{-\varepsilon_1|G_2| - \varepsilon_3|G_2|}(1 - o(\Gamma_i\gamma_i))$. Hence, by Lemma 5.3 and Lemma 5.4,

$$\Pr(\mathcal{I}_c(\emptyset, G \cap \text{NotTraversed}_i, R) \wedge \mathcal{S}_c(\emptyset, G \cap \text{NotTraversed}_i, R) \wedge |G \cap \text{Bite}_{i+1}| = 0) \geq \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right) (1 - 91\Gamma_i\gamma_i)$$

and

$$\Pr(\mathcal{I}_c(G_2, G_3, R) \wedge \mathcal{S}_c(G_2, G_3, R) \wedge G_2 \subseteq \text{Bite}_{i+1} \wedge |G_3 \cap \text{Bite}_{i+1}| = 0) \geq n^{-\varepsilon_1|G_2| - \varepsilon_3|G_2|} \left(\frac{\Phi((i+1)n^{-\varepsilon_1}) - \Phi(in^{-\varepsilon_1})}{n^{-\varepsilon_1}\phi(in^{-\varepsilon_1})} \right)^{|G_2|} \left(\frac{\phi((i+1)n^{-\varepsilon_1})}{\phi(in^{-\varepsilon_1})} \right)^{|G_3|} (1 - 91\Gamma_i\gamma_i).$$

Also, from (B3), (D3) and Claim 7.7, it follows that the number of triangles in $Y(T)$ is at least $y_{i,1,t}(1 - 100\Gamma_i - 2\Gamma_i\gamma_i) - |Z_i(R, T)|$, the number of triples (G_1, G_2, G_3) in $Y(T)$ which belong to $Y'_{i,2,1}(T)$ is at least $2n^{\varepsilon_3 - 2/5}y_{i,2,t}(1 - 100\Gamma_i - 2\Gamma_i\gamma_i)$, and the number of triples (G_1, G_2, G_3) in $Y(T)$ which belong to $Y'_{i,3,2}(T)$ is at least $3n^{2(\varepsilon_3 - 2/5)}y_{i,3,t}(1 - 100\Gamma_i - 2\Gamma_i\gamma_i)$. The claim now follows using linearity of expectation. \blacksquare

Claim 7.9. *The probability (over the choice of BigBite_{i+1} , Bite_{i+1} , and the choice of the birthtimes of the edges in Bite_{i+1}) that W_5 deviates from its expectation by more than $\Gamma_i\gamma_i y_{i+1,1,t}$ is at most $n^{-\omega(s)}$.*

Proof. For an edge $f \in \text{BIGBite}_{i+1}$, let $\text{Triangles}(f)$ be the set which contains every triangle $G \in Y(T)$ for which it holds that f has the potential of being a label in a tree in the forest $\{\mathfrak{T}_c(g, R) : g \in G \cap \text{NotTraversed}_i\}$, and every triple $(G_1, G_2, G_3) \in Y(T)$ for which it holds that f has the potential of being a label in a tree in the forest $\{\mathfrak{T}_c(g) : g \in G_2\} \cup \{\mathfrak{T}_c(g, R) : g \in G_3\}$. Observe that changing the outcome of an edge $f \in \text{BIGBite}_{i+1}$ can change W_5 by at most an additive factor of $|\text{Triangles}(f)|$.

We now use (P1), (P2) and (P3), together with (14), (15) and (16), to find that for every $f \in \text{BIGBite}_{i+1}$,

$$|\text{Triangles}(f)| \leq 3 \cdot |\text{Roots}_c(f)| + 3 \cdot |\text{Roots}_c(f) \cap \text{BIGBite}_{i+1}| \cdot 2n^{1.1/5} \leq 4n^{2.1/5}.$$

Moreover, by (13) we have that for every triangle $G \in Y(T)$, there are at most $n^{0.01}$ edges $f \in \text{BIGBite}_{i+1}$ such that $G \in \text{Triangles}(f)$, and likewise for every triple $(G_1, G_2, G_3) \in Y(T)$, there are at most $3n^{0.01}$ edges $f \in \text{BIGBite}_{i+1}$ such that $(G_1, G_2, G_3) \in \text{Triangles}(f)$, and so

$$\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)| \leq |Y(T)| \cdot 3n^{0.01} \leq n^{5.1/5},$$

where the second inequality follows from the definition of $Y(T)$ and from (18).

We want to claim that $\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)|^2 = O(n^{6.95/5})$. This will allow us to complete the proof using McDiarmid's inequality, as we did in the previous cases. However, unlike the situation in the previous cases, the discussion above does not provide directly such a bound on that sum of squares. We would have to resort to a finer analysis. The first thing to note here is that if \sum_f ranges over all edges $f \in \text{BIGBite}_{i+1}$ with $|\text{Triangles}(f)| \leq n^{1.85/5}$, then using the discussion above,

$$\sum_f |\text{Triangles}(f)|^2 \leq n^{1.85/5} \cdot \sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)| \leq n^{6.95/5}.$$

Second, and this we show below, the number of edges $f \in \text{BIGBite}_{i+1}$ with $|\text{Triangles}(f)| \geq n^{1.85/5}$ is at most $n^{2.45/5}$. Therefore,

$$\sum_{f \in \text{BIGBite}_{i+1}} |\text{Triangles}(f)|^2 \leq n^{6.95/5} + n^{2.45/5} \cdot (4n^{2.1/5})^2 = O(n^{6.95/5}).$$

It now follows from McDiarmid's inequality that the probability that W_5 deviates from its expectation by more than $\Gamma_i \gamma_i y_{i+1,1,t} \geq n^{4.99/5}$ is at most $n^{-\omega(s)}$.

To finish the proof, we argue that the number of edges $f \in \text{BIGBite}_{i+1}$ with $|\text{Triangles}(f)| \geq n^{1.85/5}$ is at most $n^{2.45/5}$. To this end, assume for the sake of contradiction that this does not hold, and fix a set E_1 of $\Theta(n^{2.45/5})$ edges $f \in \text{BIGBite}_{i+1}$ with $|\text{Triangles}(f)| \geq n^{1.85/5}$.

For an edge $f \in \text{BIGBite}_{i+1}$, let $\text{Edges}_1(f)$ be the set of edges $g \in \binom{R}{2} \cap (\text{NotTraversed}_i \setminus \text{BIGBite}_{i+1})$ for which it holds that f has the potential of being a label in the tree $\mathfrak{T}_c(g, R)$ and g belongs to some triangle or triple in $\text{Triangles}(f)$. Furthermore, let $\text{Edges}_2(f)$ be the set of edges $g \in \binom{R}{2} \cap \text{BIGBite}_{i+1}$ for which it holds that f has the potential of being a label in the tree $\mathfrak{T}_c(g)$ and g belongs to some triangle or triple in $\text{Triangles}(f)$. Note that by (P3) and since $\text{Triangles}(f) \subseteq Y(T)$, every edge $g \in \text{NotTraversed}_i \setminus \text{BIGBite}_{i+1}$ belongs to at most 3 triangles and

triples in $Triangles(f)$. Also, note that by (P1), (P2), (16) and since $Triangles(f) \subseteq Y(T)$, every edge $g \in BIGBite_{i+1}$ belongs to at most $3 \cdot 2n^{1.1/5}$ triangles and triples in $Triangles(f)$. Hence, for every edge $f \in BIGBite_{i+1}$, by the definition of $Triangles(f)$, $Edges_1(f)$ and $Edges_2(f)$, we find that

$$|Triangles(f)| \leq 3 \cdot |Edges_1(f)| + 3 \cdot |Edges_2(f)| \cdot 2n^{1.1/5}.$$

In addition, for an edge $f \in BIGBite_{i+1}$, by (14),

$$|Edges_2(f)| \cdot n^{1.1/5} \leq |Roots_c(f) \cap BIGBite_{i+1}| \cdot n^{1.1/5} \leq n^{0.01} \cdot n^{1.1/5} = o(n^{1.85/5}),$$

and so for every edge $f \in E_1$,

$$3 \cdot |Edges_1(f)| \geq |Triangles(f)| - o(n^{1.85/5}) \geq 0.5n^{1.85/5}.$$

We will reach a contradiction by showing that for some $f \in E_1$, $|Edges_1(f)| = o(n^{1.85/5})$.

Define

$$E_2 := \bigcup_{f \in E_1} Edges_1(f).$$

Observe that for an edge $f \in E_1$ and for an edge $g \in Edges_1(f)$, there exists an edge $e \in BIGBite_{i+1}$ for which the following two properties hold: first, f has the potential of being a label in the tree $\mathfrak{T}_c(e)$; second, there exists a path of length two in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{R}{2}$ that completes g to a triangle, and a graph $G \in X_{0.5}(g)$ with $G \subseteq (Traversed_i \cup BIGBite_{i+1}) \setminus \binom{R}{2}$ and $e \in G$ (and so, in particular, $e \in \binom{[n]}{2} \setminus \binom{R}{2}$). Hence, by the definition of E_2 , there exists a set E of edges in $BIGBite_{i+1}$ for which the following two properties hold: first, every edge $f \in E_1$ has the potential of being a label in a tree $\mathfrak{T}_c(e)$ for some $e \in E$; second, for every edge $g \in E_2$ there exists a path of length two in $(Traversed_i \cup BIGBite_{i+1}) \cap \binom{R}{2}$ that completes g to a triangle, and a graph $G \in X_{0.5}(g)$, with $G \subseteq (Traversed_i \cup BIGBite_{i+1}) \setminus \binom{R}{2}$ and $G \cap E \neq \emptyset$ (and so, in particular, $E \subseteq \binom{[n]}{2} \setminus \binom{R}{2}$). By (14), $|E| \leq |E_1| \cdot n^{0.01} = O(n^{1/2})$. Hence, by (C8), $|E_2| = O(n^{4.2/5})$. Thus, by (13),

$$\sum_{f \in E_1} |Edges_1(f)| \leq |E_2| \cdot n^{0.01} = O(n^{4.25/5}).$$

Therefore, there exists an edge $f \in E_1$ for which $|Edges_1(f)| = O(n^{4.25/5})/|E_1|$. Since by assumption $|E_1| = \Theta(n^{2.45/5})$, we get that there exists an edge $f \in E_1$ for which $|Edges_1(f)| = o(n^{1.85/5})$. This gives the desired contradiction. \blacksquare

8 Proof of Theorem 1.3

We begin with the following definitions. Let $0 \leq i < I$. For an integer $1 \leq j \leq 5$ and an edge $f \in NotTraversed_i$, let $X_{i,j}'''(f)$ be the set of all $G \in X_{i,j}(f)$ such that $G \subseteq M_i \cup Bite_{i+1}$. Consider the undirected graph whose vertex set is the family of all edges in $Bite_{i+1}$, and whose edge set is the family of all sets $\{g_1, g_2\}$ such that $g_2 \in G$ for some $G \in \bigcup_{1 \leq j \leq 5} X_{i,j}'''(g_1)$ (or equivalently,

$g_1 \in G$ for some $G \in \bigcup_{1 \leq j \leq 5} X_{i,j}'''(g_2)$. Let \mathcal{F}_i be the event that the size of the largest connected component in that graph has size $O_{\varepsilon_1}(1)$ (where the subscript ε_1 means, as usual, that the hidden constant depends on ε_1).

To understand the motivation behind the above definitions, we make the following observation. Let $0 \leq i < I$. Assume that we are given M_i , $Bite_{i+1}$, and a set $Triangles$ of triangles, each having two edges in M_i and one edge in $Bite_{i+1}$ which can be added to M_i without creating a copy of K_4 . Further assume that for every triangle $G \in Triangles$, the edge in $G \cap Bite_{i+1}$ belongs to exactly one triangle in $Triangles$. Lastly assume that \mathcal{F}_i holds. Then the following three properties hold: first, the event that a triangle in $Triangles$ is contained in M_{i+1} depends only on the birthtimes of $O_{\varepsilon_1}(1)$ edges in $Bite_{i+1}$; second, and this follows from the first property, given any triangle in $Triangles$, the probability that this triangle is contained in M_{i+1} is $\Omega_{\varepsilon_1}(1)$; third, changing the birthtime of a single edge in $Bite_{i+1}$ can change the number of triangles in $Triangles$ that are contained in M_{i+1} by at most an additive factor of $O_{\varepsilon_1}(1)$. Later, this observation will be used to prove Theorem 1.3. For now, we prove the following.

Lemma 8.1. *For $0 \leq i < I$, $\Pr(\mathcal{F}_i) \geq 1 - n^{-1}$.*

Proof. Fix $0 \leq i < I$. An (f, m) -cluster is a sequence $(G_l)_{l=1}^m$ such that for all $1 \leq l \leq m$ the following holds: $G_l \in X_{0,5}(g)$ for some $g \in \{f\} \cup \bigcup_{k < l} G_k$, and G_l shares at most four edges with $\{f\} \cup \bigcup_{k < l} G_k$. Say that an (f, m) -cluster $(G_l)_{l=1}^m$ is bad, if for all $1 \leq l \leq m$, $G_l \subseteq Traversed_i \cup Bite_{i+1}$ and $|G_l \cap Bite_{i+1}| \geq 1$. It should be clear that if there exists an integer m such that for every $f \in NotTraversed_i$ there is no bad (f, m) -cluster, then the largest connected component in the graph the underlies the definition of the event \mathcal{F}_i has size at most $6m$. Thus, by the union bound and by Markov's inequality, it suffices to prove that for a fixed $f \in NotTraversed_i$ and for some $m = O_{\varepsilon_1}(1)$, the expected number of bad (f, m) -clusters is at most n^{-4} .

Fix an edge $f \in NotTraversed_i$. For an (f, m) -cluster $(G_l)_{l=1}^m$, we say that G_l is a j -type if G_l shares exactly j edges with $\{f\} \cup \bigcup_{k < l} G_k$. We further say that $(G_l)_{l=1}^m$ is an $(a_0, a_1, a_2, a_3, a_4)$ -type if the number of graphs G_l of j -type is a_j . Fix a sufficiently large integer $m = O_{\varepsilon_1}(1)$. Fix a set of integers $\{a_j : 0 \leq j \leq 4\}$, so that $\sum_{j=0}^4 a_j = m$. Let a be the vector $(a_0, a_1, a_2, a_3, a_4)$. The number of a -type (f, m) -clusters of length m is trivially at most $O_{\varepsilon_1}(n^{2a_0+a_1+a_2})$. Note that for any edge $g \in \binom{[n]}{2}$, we have $\Pr(g \in Traversed_i \cup Bite_{i+1}) \leq q_1 := 2n^{\varepsilon_1^2-2/5}$ and $\Pr(g \in Bite_{i+1}) \leq q_2 := 2n^{-\varepsilon_1-2/5}$. Therefore, the probability that an a -type (f, m) -cluster is bad is at most $(q_1^4 q_2)^{a_0} \cdot q_1^{4a_1+3a_2+2a_3+a_4}$. It follows that the expected number of bad a -type (f, m) -clusters is at most $n^{-\Omega(\varepsilon_1 m)}$, which is at most n^{-5} , as m is sufficiently large. Since there are at most $O_{\varepsilon_1}(1)$ choices for the vector a , a union bound argument completes the proof. \blacksquare

We turn to prove Theorem 1.3. We need to show that a.a.s., every set $S \subseteq [n]$ of s vertices of M_I spans a triangle. In other words, letting $\exists S$ stand for “there exists a set $S \subseteq [n]$ of s vertices,” and letting $K_3 \not\subseteq M_i \cap \binom{[n]}{2}$ stand for “ $M_i \cap \binom{[n]}{2}$ is triangle-free,” we need to show that $\Pr(\exists S : K_3 \not\subseteq M_I \cap \binom{[n]}{2}) = o(1)$. Say that the process behaves if for every $0 \leq i < I$, $\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_i \wedge \mathcal{F}_i$ holds and M_i has maximum degree at most $0.01s$. From Lemma 3.2, Lemma 8.1 and a result of

Bohman and Keevash [3, Theorem 1.6] it follows that $\Pr(\text{process behaves}) = 1 - o(1)$. Therefore,

$$\begin{aligned} \Pr(\exists S : K_3 \not\subseteq M_I \cap \binom{S}{2}) &\leq \Pr(\exists S : K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) + \Pr(\neg(\text{process behaves})) \\ &= \Pr(\exists S : K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) + o(1). \end{aligned}$$

Thus, it remains to show that $\Pr(\exists S : K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) = o(1)$, and so by the union bound, it remains to fix a set $S \subseteq [n]$ of s vertices, and show that

$$\Pr(K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) = o(n^{-s}).$$

For that, we use the following lemma, whose proof is given below.

Lemma 8.2. *For $1 \leq i < I$,*

$$\Pr(K_3 \not\subseteq M_{i+1} \cap \binom{S}{2} \mid \text{process behaves} \wedge K_3 \not\subseteq M_i \cap \binom{S}{2}) \leq \exp\left(-\Omega_{\varepsilon_1}\left(n^{-\varepsilon_1-2/5}y_{i,1,s^3}\right)\right).$$

Let $I_0 := \lfloor n^{\varepsilon_1+0.5\varepsilon_1^2} \rfloor$ and recall that $I = \lfloor n^{\varepsilon_1+\varepsilon_1^2} \rfloor$. By Lemma 8.2,

$$\begin{aligned} \Pr(K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) &\leq \prod_{1 \leq i < I} \exp\left(-\Omega_{\varepsilon_1}\left(n^{-\varepsilon_1-2/5}y_{i,1,s^3}\right)\right) \\ &\leq \prod_{I_0 \leq i < I} \exp\left(-\Omega_{\varepsilon_1}\left(n^{-\varepsilon_1-2/5}y_{i,1,s^3}\right)\right) \\ &= \exp\left(-\Omega_{\varepsilon_1}(1) \cdot \sum_{I_0 \leq i < I} n^{-\varepsilon_1-2/5}y_{i,1,s^3}\right). \end{aligned}$$

Also, for every $I_0 \leq i < I$, by the definition of $y_{i,1,s^3}$ and by Lemma 3.1,

$$n^{-\varepsilon_1-2/5}y_{i,1,s^3} \geq s^3 n^{-\varepsilon_1-6/5} \Phi(in^{-\varepsilon_1})^2 \phi(in^{-\varepsilon_1}) = \Omega\left(\frac{C^2 s}{i}\right).$$

Therefore,

$$\begin{aligned} \Pr(K_3 \not\subseteq M_I \cap \binom{S}{2} \mid \text{process behaves}) &\leq \exp\left(-\Omega_{\varepsilon_1}(1) \cdot C^2 s \cdot \sum_{I_0 \leq i < I} \frac{1}{i}\right) \\ &\leq \exp\left(-\Omega_{\varepsilon_1}(1) \cdot C^2 s \ln n\right). \end{aligned}$$

Taking $C = C(\varepsilon_1)$ sufficiently large, the last bound is at most $o(n^{-s})$. This gives Theorem 1.3.

8.1 Proof of Lemma 8.2

Fix $1 \leq i < I$. We want to bound the probability that $M_{i+1} \cap \binom{S}{2}$ is triangle-free, conditioned on the event that the process behaves and that $M_i \cap \binom{S}{2}$ is triangle-free. For that purpose, assume that we are given M_i so that $\mathcal{A}_i \wedge \mathcal{B}_i \wedge \mathcal{C}_{i-1}$ holds, M_i has maximum degree at most $0.01s$, and $M_i \cap \binom{S}{2}$ is triangle-free. Given that assumption, we bound the probability that $M_{i+1} \cap \binom{S}{2}$ is triangle-free (where the probability is over the choice of $BIGBite_{i+1}$, $BigBite_{i+1}$, $Bite_{i+1}$ and the choice of the birthtimes of the edges in $Bite_{i+1}$), conditioned on the event that the process behaves.

Let $S_i \subseteq S$ be the set that is guaranteed to exist by \mathcal{B}_i .

Claim 8.3. *There exists a pair $(R, T) \in \text{Pairs}(S_i)$ such that $|Y_{i,1}(T)| = \Omega(y_{i,1,s^3})$.*

Proof. By (B3) and Lemma 3.1, for every pair $(R, T) \in \text{Pairs}(S_i)$ with $|T| = t$, we have $|Y_{i,1}(T)| = \Omega(y_{i,1,t}) - |Z_i(R, T)|$, and furthermore, since $t = \Omega(s^3)$, we have $y_{i,1,t} = \Omega(y_{i,1,s^3})$. Hence, for every pair $(R, T) \in \text{Pairs}(S_i)$,

$$|Y_{i,1}(T)| = \Omega(y_{i,1,s^3}) - |Z_i(R, T)|.$$

We show below that for some pair $(R, T) \in \text{Pairs}(S_i)$, we have $|Z_i(R, T)| = O(n^{4.2/5})$. This, together with Lemma 3.1, will imply that $|Z_i(R, T)| = o(y_{i,1,s^3})$ and so the claim will follow.

Consider an arbitrary pair $(R, T) \in \text{Pairs}(S_i)$. Recall that $Z_i(R, T)$ is the set of all triangles $G \in T$ such that $|M_i \cap G| = 2$, $|\text{NotTraversed}_i \cap G| = 1$, and letting g denote the edge in $\text{NotTraversed}_i \cap G$, there exists $G_0 \in X_{i,0}(g)$ such that G_0 shares at least three vertices with R . Since $M_i \subseteq \text{Traversed}_{i-1} \cup \text{BIGBite}_i$, since $\mathcal{C}_{i-1}(\text{C6})$ holds and since $R \subseteq S$, there is a set E_0 of at most $n^{3.1/5}$ edges, the removal of which from M_i leaves at most $n^{4.2/5}$ 4-cycles in $M_i \cap \binom{R}{2}$. In addition, since $M_i \subseteq \text{Traversed}_i$, since (B2) holds and since $R \subseteq S_i$, every edge in E_0 belongs to at most $2n^{1.1/5}$ triangles in $Z_i(R, T)$. Hence, except for at most $4n^{4.2/5} + |E_0| \cdot 2n^{1.1/5} = O(n^{4.2/5})$ triangles, for every triangle $G \in Z_i(R, T)$, we have that the edge $g \in \text{NotTraversed}_i \cap G$ belongs to no other triangle in $Z_i(R, T)$. Therefore, up to an additive factor of $O(n^{4.2/5})$, the number of triangles in $Z_i(R, T)$ is at most the number of edges $g \in \binom{R}{2}$ that belong to some triangle in T , and for which there exists $G_0 \in X_{i,0}(g)$ such that G_0 shares at least three vertices with R .

Since $M_i \subseteq \text{Traversed}_{i-1} \cup \text{BIGBite}_i$, since $\mathcal{C}_{i-1}(\text{C7})$ holds and since $s - o(s) \leq |S_i| \leq |S| \leq s$ by (B1), assuming the maximum degree in $M_i \cap \binom{S_i}{2}$ is at most $n^{1.1/5}$, S_i satisfies the following two properties. First, there are at most $O(n^{4.2/5})$ edges $g \in \binom{S_i}{2}$ for which there exists a graph $G_0 \in X_{i,0}(g)$, which shares all four vertices with S_i . Second, there is a set $R_0 \subseteq [n] \setminus S_i$ of at most $n^{0.99/5}$ vertices (which we fix for the rest of the proof), such that there are at most $O(n^{4.2/5})$ edges $g \in \binom{S_i}{2}$ for which there exists a graph $G_0 \in X_{i,0}(g)$, which shares exactly three vertices with S_i and one vertex with $[n] \setminus (S_i \cup R_0)$. Also note that since $M_i \subseteq \text{Traversed}_i$ and since (B2) holds, the maximum degree in $M_i \cap \binom{S_i}{2}$ is at most $n^{1.1/5}$. Hence, it remains to show that for some pair $(R, T) \in \text{Pairs}(S_i)$, the number of edges $g \in \binom{R}{2}$ that belong to some triangle in T , and for which there exists $G_0 \in X_{i,0}(g)$ such that G_0 shares three vertices with R and one vertex with R_0 is at most $O(n^{4.2/5})$.

Let $V \subseteq S_i$ be the set of vertices $v \in S_i$ such that v is adjacent in M_i to at most one vertex in R_0 . We have the following two observations. The first observation is that since $M_i \subseteq \text{Traversed}_{i-1} \cup \text{BIGBite}_i$ and since $\mathcal{C}_{i-1}(\text{C3})$ holds, $|S_i \setminus V| \leq |R_0|^2 \cdot n^{1/5+10\epsilon_3} = o(s)$, and so $|V| \geq s - o(s)$. The second observation is that by this lower bound on the size of V and since M_i has maximum degree at most $0.01s$, there exists a partition of the vertices of V to three sets of vertices, each of size $\Omega(s)$, such that there is no vertex in R_0 that is adjacent in M_i to two vertices in two different parts of the partition. Fix such a partition and consider the pair $(V, T) \in \text{Pairs}(S_i)$ that corresponds to that partition. Then there is no triangle in T with an edge whose two vertices are adjacent in M_i to a vertex in R_0 . Hence, the number of edges $g \in \binom{V}{2}$ that belong to some triangle in T , and for which there exists $G_0 \in X_{i,0}(g)$ such that G_0 shares three vertices with V and one vertex with R_0 is 0. This completes the proof. \blacksquare

Fix for the rest of the proof a pair $(R, T) \in \text{Pairs}(S_i)$, as guaranteed to exist by the above claim, so that $Y_{i,1}(T) = \Omega(y_{i,1,s^3})$. Let \mathcal{E} be the event that there exists a set $\text{Triangles} \subseteq Y_{i,1}(T)$ with the following three properties: first, every triangle in Triangles has two edges in M_i and one edge in Bite_{i+1} (which can be added to M_i without creating a copy of K_4); second, for every triangle $G \in \text{Triangles}$, the edge in $G \cap \text{Bite}_{i+1}$ belongs to exactly one triangle in Triangles ; third, $|\text{Triangles}| = \Omega(n^{-\varepsilon_1-2/5}y_{i,1,s^3})$. We have

$$\begin{aligned} & \Pr(K_3 \not\subseteq M_{i+1} \cap \binom{S}{2} \mid \text{process behaves} \wedge K_3 \not\subseteq M_i \cap \binom{S}{2}) \leq \\ & \Pr(K_3 \not\subseteq M_{i+1} \cap \binom{S}{2} \mid \mathcal{E} \wedge \text{process behaves} \wedge K_3 \not\subseteq M_i \cap \binom{S}{2}) + \\ & \Pr(\neg \mathcal{E} \mid \text{process behaves} \wedge K_3 \not\subseteq M_i \cap \binom{S}{2}). \end{aligned}$$

We bound each of the two last terms by $\exp(-\Omega_{\varepsilon_1}(n^{-\varepsilon_1-2/5}y_{i,1,s^3}))$.

To bound the first term, it is enough to bound the probability of the event $K_3 \not\subseteq M_{i+1} \cap \binom{S}{2}$, under the assumption that we are given M_i and Bite_{i+1} , \mathcal{E} holds, the process behaves, and $K_3 \not\subseteq M_i \cap \binom{S}{2}$. (The probability here is over the choice of the birthtimes of the edges in Bite_{i+1} .) Under this assumption, we can use the observation that was given at the beginning of the section to claim the following. First, the event that a triangle in Triangles is contained in M_{i+1} depends only on the birthtimes of $O_{\varepsilon_1}(1)$ edges in Bite_{i+1} . Second, the expected number of triangles in Triangles that are contained in M_{i+1} is $\Omega_{\varepsilon_1}(|\text{Triangles}|)$, which is $\Omega_{\varepsilon_1}(n^{-\varepsilon_1-2/5}y_{i,1,s^3})$. Third, changing the birthtime of a single edge in Bite_{i+1} can change the number of triangles in Triangles that are contained in M_{i+1} by at most an additive factor of $O_{\varepsilon_1}(1)$. Therefore, the probability of the event $K_3 \not\subseteq M_{i+1} \cap \binom{S}{2}$ is at most the probability that no triangle in Triangles is in M_{i+1} , which given the assumptions and the three claims above, by McDiarmid's inequality, is at most $\exp(-\Omega_{\varepsilon_1}(n^{-\varepsilon_1-2/5}y_{i,1,s^3}))$, as needed.

Next, we bound the second term. We claim that under the assumption that we are given M_i so that the process behaves and $K_3 \not\subseteq M_i \cap \binom{S}{2}$ holds, $\neg \mathcal{E}$ occurs with probability at most $\exp(-\Omega(n^{-\varepsilon_1-2/5}y_{i,1,s^3}))$. (The probability here is over the choice of BIGBite_{i+1} , BigBite_{i+1} and Bite_{i+1} .) To prove that claim, first recall that $|Y_{i,1}(T)| = \Omega(y_{i,1,s^3})$. Next, using an argument similar to the one used in the proof of Claim 8.3, one can find that there is a set $Y_{i,1}^*(T) \subseteq Y_{i,1}(T)$ of size $\Omega(|Y_{i,1}(T)|)$, that is of size $\Omega(y_{i,1,s^3})$, such that for every triangle $G \in Y_{i,1}^*(T)$, the edge in $\text{NotTraversed}_i \cap G$ belongs to exactly one triangle in $Y_{i,1}^*(T)$. In particular, every triangle $G \in Y_{i,1}^*(T)$ is uniquely determined by the edge in $\text{NotTraversed}_i \cap G$. Consider the set of $\Omega(y_{i,1,s^3})$ edges that determine the triangles in $Y_{i,1}^*(T)$, and note that by Chernoff's bound, $\Omega(n^{-\varepsilon_1-2/5}y_{i,1,s^3})$ edges of these are in Bite_{i+1} with probability at least $1 - \exp(-\Omega_{\varepsilon_1}(n^{-\varepsilon_1-2/5}y_{i,1,s^3}))$. So with probability at least $1 - \exp(-\Omega_{\varepsilon_1}(n^{-\varepsilon_1-2/5}y_{i,1,s^3}))$, there is a set of $\Omega(n^{-\varepsilon_1-2/5}y_{i,1,s^3})$ triangles in $Y_{i,1}^*(T)$, each of which has two edges in M_i and one edge in Bite_{i+1} . This completes the proof.

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